MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LIX.

MATHEMATICAL

QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN THE "EDUCATIONAL TIMES,"

AND

AN APPENDIX.

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR OF THE GENERAL MEDICAL COUNCIL.

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Solved Questions.

APPENDIX.

MATHEMATICS

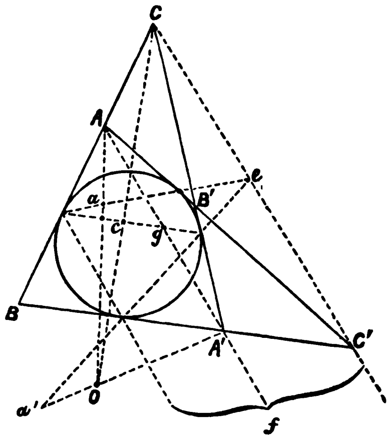
FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

11740. (J. W. Russell, M.A.)—The opposite vertices AA′, BB′, CC′ of a quadrilateral circumscribing a conic are joined to a given point O; OA cuts the polar of A in a, OB cuts the polar of B in b, and so on; show that a conic can be drawn through the seven points O, a, a', b, b', c, c'.

Solution by J. C. St. Clair; M. Brierley; *and others.*



Let e, f, g be the diagonal points of the quadrangle whose sides are the polars of A, B, C, &c.

The harmonic pencil $(e\,.\,aga'f)$ = the polar range $(AfA'g)$; therefore

$$(e\,.\,aga'f) = (O\,.\,AgA'f) = (O\,.\,aga'f).$$

Hence the six points $O\,e\,f\,g\,a\,a'$ lie on the same conic (1).

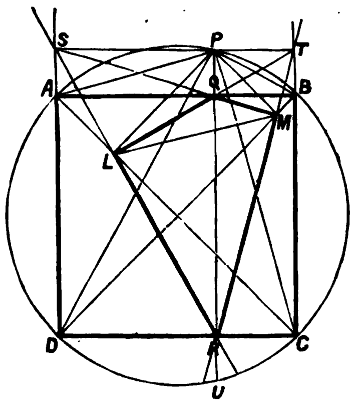
Similarly, $O\,e\,f\,g\,b\,b'$ lie on a conic (2); and $O\,e\,f\,g\,c\,c'$ lie on a conic (3).

Again, since the pairs Oa, fg; Oc, ef; ae, cg meet in the three collinear points A, C, and the point of contact of AC, $Oaefgc$ is a Pascal hexagon; and therefore the conic (1) is identical with (3). It may be shown in the same manner that $(1) \equiv (2)$. Hence the six points $a\,a'\,b\,b'\,c\,c'$ and also $O\,e\,f\,g$ lie on the same conic.

[The Proposer's solution is as follows:—Project O into the centre. Then $a\,a'\,b\,b'\,c\,c'$ are the middle points of the six sides of the quadrangle L M N R inscribed in the conic, where L M N R are the points of contact of the circumscribed quadrilateral. Hence O the centre lies on the centre-locus, viz., the conic through $a\,a'\,b\,b'\,c\,c'$.]

7373. (D. Edwardes.)—ABCD is a square inscribed in a circle, and P a point on the circumference of the circle. The pedal lines with respect to P are drawn of the triangles formed by the sides and diagonals of ABCD. Prove (1) that the area of the quadrilateral formed by these pedal lines is $\frac{1}{4}(PA.PC+PB.PD)$; (2) its maximum area $\frac{1}{4}AB^2\sqrt{2}$, and (3) the angle between its diagonals is $\sin^{-1}\left(\frac{PL+PM}{AB}\right)$ where PL, PM are the perpendiculars from P on the diagonals of the square.

Solution by Professor Schoute.



1. The three points S, Q, L lie on the circle of which AP is a diameter (for the angles ASP, AQP, ALP are right ones). This gives

$$\angle QSL = \angle QAL = 45^\circ,$$
$$\angle QLS = \angle QAS = 90^\circ;$$
$$\therefore\ QL = \tfrac{1}{2}QS\sqrt{2} = \tfrac{1}{2}AP\sqrt{2}.$$

Also, $QM = \frac{1}{2}BP\sqrt{2}$, $LR = \frac{1}{2}CP\sqrt{2}$, $MR = \frac{1}{2}DP\sqrt{2}$, $\angle LRM = 45^\circ$, $\angle MQL = 135^\circ$;

$\therefore\ \triangle QLR \sim \triangle APC$, and area $QLR = \frac{1}{2}\triangle APC$, $\triangle QMR \sim \triangle BPD$, and area $QMR = \frac{1}{2}\triangle BPD$,

area $QLRM = \frac{1}{2}\triangle APC + \frac{1}{2}\triangle BPD = \frac{1}{4}(PA.PC+PB.PD.)$

2. We have $\triangle QLM \sim \triangle PAB$, and area $QLM = \frac{1}{2}\triangle PAB$,
$\triangle RLM \sim \triangle PCD$, and area $RLM = \frac{1}{2}\triangle PCD$.

So area $QLRM = \frac{1}{2}\triangle PAB + \frac{1}{2}\triangle PCD = \frac{1}{4}AB(PQ+PR) = \frac{1}{4}AB.PU$.

Now PU is a maximum ($= AB\sqrt{2}$), if P bisects the arc AB. Then area $QLRM = \frac{1}{4}AB^2\sqrt{2}$.

3. If we project the lines PL + PM, and LM on AB, we find

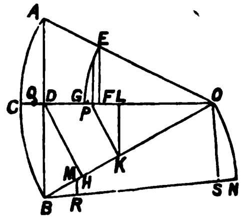
$$(PL+PM)\tfrac{1}{2}\sqrt{2} = LM\sin\alpha,$$

where α is the angle of the diagonals LM, QR. Now $LM = \frac{1}{2}AB\sqrt{2}$. Thus $\sin\alpha = \frac{PL+LM}{AB}$. In another way we find $\sin\alpha = \frac{PU}{AB\sqrt{2}}$, a result that also proves the second part.

11729. (Professor Orchard, M.A., B.Sc.)—Show how to deduce the position of the centre of gravity of a segment of a circle from that of a sector, and *vice versâ*.

Solution by D. Biddle; Professor Zerr; *and others.*

Let OAB be the sector, $OA = 1$, $\angle AOB = 2\alpha$. Bisect AB in D, and draw OC through D. Then $\angle AOC = \alpha$. Make $OE = \frac{2}{3}OA$, and draw EF perpendicular to OC. Then F is the centre of gravity of the triangle ABO. Also describe the arc EGE′ between OA and OB from centre O. Then the centre of gravity of that arc ($\equiv$ P) is also the centre of gravity of the sector. Moreover, the area of the triangle



$$ABO = \sin\alpha \,.\, \cos\alpha,$$

and the area of the sector $= \alpha$. Also $OF = \frac{2}{3}\cos\alpha$, and $OP = \frac{2}{3}\sin\alpha/\alpha$. Let Q be the centre of gravity of the segment ABC.

Then $FP : PQ = \text{area ACB} : \text{area ABO} = \alpha - \sin\alpha \,.\, \cos\alpha : \sin\alpha \,.\, \cos\alpha$

$$= \tfrac{2}{3}\left(\frac{\sin\alpha}{\alpha} - \cos\alpha\right) : PQ,$$

whence $$PQ = \tfrac{2}{3}\left(\frac{\sin^2\alpha\cos\alpha}{\alpha} - \sin\alpha\cos^2\alpha\right) \Big/ (\alpha - \sin\alpha \,.\, \cos\alpha),$$

and $$OQ = PQ + OP = OP \,.\, \sin^2\alpha/(1 - \tfrac{3}{2}OP \,.\, \cos\alpha).$$

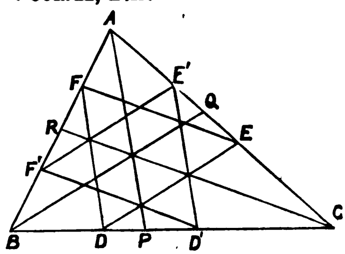
Moreover, $$OP = OQ/(\sin^2\alpha + \tfrac{3}{2}OQ \,.\, \cos\alpha).$$

In order to apply this geometrically, draw the perpendiculars DH, PK on OB, and KL on OC. Then $BD = \sin\alpha$, $BH = \sin^2\alpha$. Similarly $PK = OP \,.\, \sin\alpha$, $PL = OP \,.\, \sin^2\alpha$, and $OK = OP \,.\, \cos\alpha$. Make $KM = \frac{1}{2}OK$. Then $BM = 1 - \frac{3}{2}OP \,.\, \cos\alpha$. Draw $BN = BO$, and make $BR = PL$. Join MR and draw OS parallel to it. Then $BS = OQ$, and $NS = CQ$. Thus Q is found from P, and the reverse is equally easy.

11706. (R. Tucker, M.A.)—DEF, D′E′F′ are in-triangles of ABC (D, D′ on BC; E, E′ on CA; F, F′ on AB), which have their sides parallel to the bisectors of the angles of ABC. Find the areas of the triangles and the equations to their circumcircles, and show that DD′EE′FF′ is an equilateral hexagon having each side equal $abc/\Sigma(ab)$.

Solution by H. W. Curjel, B.A.

Let the bisectors of the angles A, B, C meet the opposite sides in P, Q, R: then



$$CQ : QE = CB : BD,$$
$$BP : PD = BA : AF,$$
$$AR : RF = AC : CE;$$

i.e., $$ba/(a+c) : ba/(a+c) - CE = a : BD,$$

$$ca/(b+c) : ca/(b+c) - BD = c : AF,$$

$$cb/(b+a) : cb/(b+a) - \mathrm{AF} = b : \mathrm{CE}.$$

Hence $\mathrm{CE} = \dfrac{b \, . \, ba}{ba+ca+cb}$, $\mathrm{AE'} = \dfrac{b \, . \, bc}{ab+bc+ca}$, $\mathrm{CD'} = \dfrac{a \, . \, ab}{ab+bc+ca}$;

therefore $$\frac{\mathrm{CE}}{\mathrm{CA}} = \frac{ab}{ac+bc+ab} = \frac{\mathrm{CD'}}{\mathrm{CB}};$$

therefore D′E is parallel to AB; therefore

$$\angle \mathrm{D'ED} = \tfrac{1}{2}\mathrm{B} = \angle \mathrm{EDD'}; \ \therefore \ \mathrm{DD'} = \mathrm{D'E}.$$

Similarly, $\mathrm{DD'} = \mathrm{DF'} = \mathrm{F'F} = \mathrm{FE'} = \mathrm{EE'}.$

Hence DD′EE′FF′ is an equilateral hexagon with each side

$$= \mathrm{EE'} = b - \mathrm{CE} - \mathrm{AE'} = \frac{abc}{\Sigma\, bc}.$$

Hence also we see that $\mathrm{CE} : \mathrm{EE'} : \mathrm{E'A} = ab : ac : bc$, and the other sides are divided in corresponding ratios. Hence

$$\begin{aligned}\Delta\mathrm{DEF} &= \mathrm{S} - \Delta\mathrm{CED} - \Delta\mathrm{BDF} - \Delta\mathrm{AFE} \\ &= \mathrm{S}\left\{1 - \frac{ba\,(ba+bc)+ac\,(ac+ab)+bc\,(bc+ac)}{(\Sigma\, ab)^2}\right\} \\ &= \mathrm{S}\frac{abc\,(a+b+c)}{(\Sigma\, ab)^2}, \quad \text{where } \mathrm{S} = \Delta\mathrm{ABC}.\end{aligned}$$

[It has been assumed that the triangles DEF, D′E′F′ exist, but, if we divide the side AC so that $\mathrm{CE} : \mathrm{EE'} : \mathrm{E'A} = ba : ac : bc$, and the other sides in corresponding ratios, the sides of the triangles DEF, D′E′F′ are evidently parallel to the bisectors of the angles of the triangle ABC.

The Proposer's proof of his theorem is as follows:—

By successive applications of Euc. vi. 3, we readily get

$$\mathrm{BD} = a^2c/\Sigma\,(ab), \quad \mathrm{CD} = ab\,(c+a)/\Sigma\,(ab);$$
$$\mathrm{CD'} = a^2b/\Sigma\,(ab), \quad \mathrm{BD'} = ca\,(a+b)/\Sigma\,(ab);$$

hence $$\mathrm{DD'} = abc/\Sigma\,(ab) = \mathrm{EE'} = \mathrm{FF'}.$$

If AL is the bisector of angle A, then

$$\mathrm{DF} : \mathrm{AL} = \mathrm{BD} : \mathrm{BL}, \quad \mathrm{D'F'} : \mathrm{AL} = \mathrm{CD'} : \mathrm{CL},$$

and $$\mathrm{BD} : \mathrm{CD'} = c : b = \mathrm{BL} : \mathrm{CL}; \ \therefore \ \mathrm{DF} = \mathrm{D'F'}.$$

Hence DD′E′F is a parallelogram, and E′F = DD′, *i.e.*, the hexagon is equilateral.

By the geometry it is evident that DEF, D′E′F′ are congruent, the angles being $\mathrm{D} = 90^\circ - \tfrac{1}{2}\mathrm{C}$, $\mathrm{E} = 90^\circ - \tfrac{1}{2}\mathrm{A}$, $\mathrm{F} = 90^\circ - \tfrac{1}{2}\mathrm{B}$;

$$\mathrm{D'} = 90^\circ - \tfrac{1}{2}\mathrm{B}, \ \mathrm{E'} = 90^\circ - \tfrac{1}{2}\mathrm{C}, \ \mathrm{F'} = 90^\circ - \tfrac{1}{2}\mathrm{A};$$

and $$\mathrm{DF} = 2abc \cos \tfrac{1}{2}\mathrm{A}/\Sigma\,(ab) = \mathrm{D'E'}, \text{ \&c.}$$

Hence $$\Delta\mathrm{DEF} = \Delta\mathrm{D'E'F'} = abc\,.\,\Sigma\,(a)/[\Sigma\,(ab)]^2 \times \Delta\mathrm{ABC}.$$

In trilinears, the coordinates are proportional to

$$\mathrm{D}\,(0,\ c+a,\ a), \quad \mathrm{E}\,(b,\ 0,\ a+b), \quad \mathrm{F}\,(b+c,\ c,\ 0).$$

In the usual way we find equation to circle DEF to be

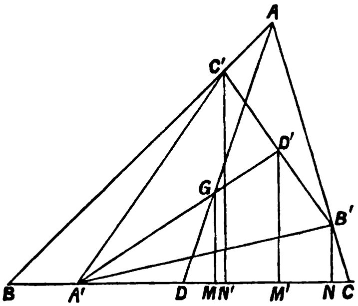
$$\Sigma(\alpha\beta\gamma) = \frac{\Sigma(a\alpha)}{(\Sigma ab)^2}\left[\Sigma\left\{a+b\,.\,c\,.\,c^2-a^2+ab\right\}\alpha\right],$$

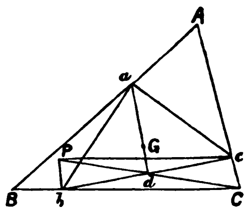
and to D′E′F′, $\Sigma(\alpha\beta\gamma) = \dfrac{\Sigma(a\alpha)}{(\Sigma ab)^2}\left[\Sigma\left\{a+c\,.\,b\,.\,b^2-a^2+ac\right\}\alpha\right].$]

11656. (Professor DESPREZ.)—On inscrit à un triangle fixe ABC tous les triangles A′B′C′ ayant même centre de gravité G. Démontrer que les côtés B′C′, C′A′, A′B′ enveloppent trois paraboles.

Solution by R. F. DAVIS; J. DALE; *and others.*

Take any point a on AB; join aG and produce it to d, so that Gd = ½aG. Produce Cd to P so that Cd = dP, and draw Pb, Pc parallel to CA, CB meeting CB, CA in b, c respectively. Then the diagonals bc,





CP of the parallelogram PbCc bisect each other in d. Thus a triangle abc has been inscribed to the triangle ABC, having the same centre of gravity G.

The locus of d is obviously a fixed straight line parallel to AB; whence so also is the locus of P. Hence the envelope of bc is a parabola having CA, CB for tangents, and the locus of P for the corresponding chord of contact. (See MILNE and DAVIS, "The Parabola," p. 25, Prop. XVIII.)

11422. (D. BIDDLE.)—Of $2n$ trees, planted in a row, alternate ones are cut down; and the process is annually repeated with those remaining, until only one is left, odds and evens being cut down by turns in the successive sets, counted from the same end. Find, in terms of n, the position (in the original belt) of the last tree left, (1) when odds begin, (2) when evens begin.

Solution by H. J. Woodall, A.R.C.S.

(1) It can be easily seen that the series giving the first to be cut down at the successive operations is 1, 4, 2, 14, 6, 54, 22 = u_1, u_2, &c., the first left being 1, 2, 2, 6, 6, &c. = v_1, v_2, v_3, &c.; $u_1 = 1$, $u_2 = 4$, $u_3 = 2$, &c.; therefore $u_2 - u_1 = 3$ and $v_2 = v_3 = u_3$; $u_2 - u_3 = 2$;

$$u_4 - u_3 = 12 = 4 \times 3 = 3 \times 2^2; \quad \therefore \; u_{2m} - u_{2m-1} = 3 \times 2^{2m-2};$$

$$u_4 - u_5 = 8 = 4 \times 2 = 2^3, \qquad u_{2m} - u_{2m+1} = 2^{2m-1},$$

Thus $\quad u_{2m} = u_{2m+1} + 2^{2m-1}, \quad u_{2m} = u_{2m-1} + 3 \times 2^{2m-2};$

therefore $\quad u_{2m+1} - u_{2m-1} = 3 \times 2^{2m-2} - 2^{2m-1} = 2^{2m-2}.$

Also $\quad u_{2m+1} = 1 + \Sigma_1^m 2^{2x-2} = (2^{2m} - 1)/(2^2 - 1) + 1$

$$= \tfrac{1}{3}(2^{2m} + 2) = v_{2m+1} = v_{2m},$$

and $\quad u_{2m} = \tfrac{1}{3}(2^{2m} + 2) + 2^{2m-1}.$

(2) In this case the series of u's is 2, 1, 7, 3, 27, 11, 107, 43, 299, 171,

" " v's is 1, 1, 3, 3, 11, 11, 43, 43,

and we find, as before,

$$u_{2m} = \tfrac{1}{3}(2^{2m-1} + 1) = v_{2m-1} = v_{2m}, \quad u_{2m+1} = \tfrac{1}{3}(2^{2m-1} + 1) + 3 \times 2^{2m-1}.$$

[The Proposer observes that, admirable as is the above elucidation of an intricate question, the result is not given in terms of n, as desired. Nor is this easy, for obviously the last tree (k) left standing may be the same for several values of n. Thus, in (1), if $2n = 2$ or 4, $k = 2$; if $2n = 6, 8, 10, 12, 14, 16, 18$, or 20, $k = 6$. But the following holds good for (1); if

$$\{1 + \tfrac{1}{3}(2^{2m} - 1)\} \not> 2n < \{1 + \tfrac{1}{3}(2^{2m+2} - 1)\},$$

then $k = 1 + \tfrac{1}{3}(2^{2m} - 1)$, the first of the terms. By simplification, this may be rendered, if

$$2^{2x} \not> (6n - 2) < 2^{2x+2}, \quad k = 1 + \tfrac{1}{3}(2^{2x} - 1);$$

x can be found by continuous division of $(6n - 2)$ by 2, until further division would reduce the quotient below 1. The number of divisions so made $= 2x$ or $2x + 1$. Thus, let $2n = 44$, then $6n - 2 = 130$, and $130 \div 2 = 65$, $\div 2 = 32\frac{1}{2}$, $\div 2 = 16\frac{1}{4}$, $\div 2 = 8\frac{1}{8}$, $\div 2 = 4\frac{1}{16}$, $\div 2 = 2\frac{1}{32}$, $\div 2 = 1\frac{1}{64}$, in all, 7 divisions. Therefore $x = 3$, and $k = 22$.

In (2), the successive leading trees left are those of (1) divided by 2, namely, 1, 3, 11, &c., instead of 2, 6, 22, &c. And, if

$$\tfrac{1}{3}(2^{2m-1} + 1) \not> (2n - 1) < \tfrac{1}{3}(2^{2m+1} + 1),$$

then $k = \tfrac{1}{3}(2^{2m-1} + 1)$. Or, by simplification, if

$$2^{2y-1} \not> (6n - 4) < 2^{2y+1}, \quad k = \tfrac{1}{3}(2^{2y-1} + 1);$$

y can be found by continuous division of $6n - 4$ by 2, and the number of divisions $= 2y$ or $2y - 1$, as they are even or odd.]

11719. (Professor MORLEY.)—Given a homogeneous line equation of a curve, $f(p_1, p_2, \dots p_n) = 0$, where p_n is the distance from a fixed point a_n to a line, prove that the foci are given by $f(z-a_1, z-a_2, \dots z-a_n) = 0$.

Solution by Prof. GENESE, M.A.; Prof. ZERR, M.A.; *and others.*

Let a focus be taken as origin O; then the isotropic line $x+y(-1)^{\frac{1}{2}} = 0$ is a tangent (t). Let x_n, y_n be the coordinates of a_n. The perpendiculars from $a_1, a_2 \dots a_n$ on t are all infinite, *but are proportional to*

$$x_1+y_1(-1)^{\frac{1}{2}}, \quad x_2+y_2(-1)^{\frac{1}{2}}, \ \&c.;$$

therefore $f\left[x_1+y_1(-1)^{\frac{1}{2}}, x_2+y_2(-1)^{\frac{1}{2}}, \dots x_n+y_n(-1)^{\frac{1}{2}}\right] = 0$;

therefore, *in equipollences*, $f(0a_1, 0a_2 \dots 0a_n) = 0$;

or, changing origin, $f\left[z-a_1, z-a_2 \dots (z-a_n)\right] = 0$,

where z is the vector of any focus, which is Professor MORLEY's beautiful theorem. Particular cases will be found most interesting.

11683. (Rev. ROBERT BRUCE, D.D.)—Show (1) how to place eight men on a draught-board so that no two of them shall be in line with another, horizontally, perpendicularly, or diagonally; and find (2) in how many ways this can be done.

Solution by the PROPOSER.

The question arose while I was staying at Windermere. Dr. PARKER, who was there, had "a puzzle" which he set every new arrival to solve; it was the one in the question. Not much difficulty was found in placing seven; but the eighth was the crux. Dr. PARKER then showed us the thing done. I asked if that was the only position. He replied that he thought it could be done two ways. I began studying and experimenting, and found at least forty ways, or forty positions in which the conditions are fulfilled. By observing a certain sequence of the numbers of the places in which the men stood, I saw at once that one position virtually involves four by simply turning round the board. Taking the rows in order from the bottom to the top, I found these (*inter alia*):—

Row.	One set or position (4 ways).				Second set.				Row.
I.	4	6	6	4	1	4	6	8	I.
II.	2	3	3	7	7	2	3	4	II.
III.	7	7	1	1	5	7	5	1	III.
IV.	5	4	8	8	8	3	7	3	IV.
V.	1	1	4	5	2	6	1	6	V.
VI.	8	8	2	2	4	8	4	2	VI.
VII.	6	2	7	6	6	5	2	7	VII.
VIII.	3	5	5	3	3	1	8	5	VIII.

I said to Dr. PARKER, "I want to find out a mathematical principle underlying the puzzle." I observe that the position of one man relative to another, is, in the majority of cases, on the principle of the knight's move in chess.

I then adopted a different principle of marking the board and trying the effect of that as to whether the problem could not be solved by beginning on any spot in the board, and by experiment I believe it can, because every square will be occupied in some of the arrangements. I numbered the places thus:—

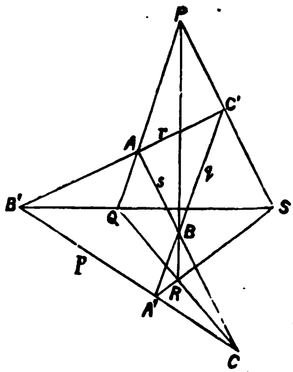
Row VIII. ...	57	58	59	60	61	62	63	64
,, VII. ...	49	50	51	52	53	54	55	56
,, VI. ...	41	42	43	44	45	46	47	48
,, V. ...	33	34	35	36	37	38	39	40
,, IV. ...	25	26	27	28	29	30	31	32
,, III. ...	17	18	19	20	21	22	23	24
,, II. ...	9	10	11	12	13	14	15	16
,, I. ...	1	2	3	4	5	6	7	8

and, having written down the places occupied, in order, I found results such as are tabulated below, and, on adding up the numbers, I found, in every case, their total was identical, 260, so that, if that number was not made, I had made some mistake. Some contributor will, I hope, explain the mathematical principle involved by the problem.

2	3	5	5	5	5	7	5	4	7	1	... I. Row.
13	14	11	16	9	15	9	15	9	10	13	... II. ,,
23	20	17	20	20	18	19	18	21	22	24	... III. ,,
25	26	30	25	30	30	32	30	32	27	30	... IV. ,,
35	40	40	39	40	35	38	35	38	33	35	... V. ,,
48	45	42	42	42	41	44	41	43	44	47	... VI. ,,
54	55	52	54	55	52	60	52	55	56	50	... VII. ,,
60	57	63	59	59	64	61	64	58	61	60	... VIII. ,,
260	260	260	260	260	260	260	260	260	260	260	

6521. (The late T. COTTERILL, M.A.)—Conicoids circumscribing a quartic curve in space have a common self-conjugate tetrahedron. Show that a plane cuts the quartic in four points, and the tetrahedron in four lines, such that triangles can be found circumscribing any conic inscribed in the four lines conjugate to any conic through the four points.

Solution by Professor SCHOUTE.



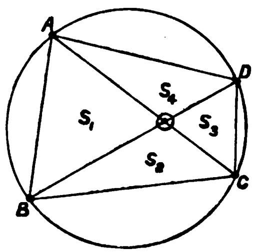
A plane Σ cuts the opposite edges of a tetrahedron PQRS in three couples of points (AA′), (BB′), (CC′). The points of each couple are conjugate with reference to the section S^2 of Σ and any quadric for which the tetrahedron is autopolar.

Now the couples (AA′), (BB′), (CC′) represent the degenerated individuals of the tangential pencil of curves of the second class touching the four lines of intersection p, q, r, s of Σ and the faces of PQRS. And the indicated relation between S^2 and each of the three couples represents the special case of the relation mentioned in the problem corresponding to the case of a curve of the second class degenerated into two points. Moreover the relation in question exists between S^2 and any conic T^2 of a tangential pencil, if it exists between S^2 and two individuals of the pencil. Therefore, &c.

11694. (Professor VUITTENEZ.)—On considère un quadrilatère inscriptible ABCD dans lequel les diagonales se coupent au point O. Si S_1, S_2, S_3, S_4 désignent respectivement les surfaces des triangles OAB, OBC, OCD, ODA, démontrer que

$$AC : BD = \{(S_1S_4)^{\frac{1}{2}} + (S_2S_3)^{\frac{1}{2}}\} : \{(S_1S_2)^{\frac{1}{2}} + (S_3S_4)^{\frac{1}{2}}\}.$$

Solution by R. F. DAVIS, M.A.; H. W. CURJEL, B.A.; *and others.*



Since the angles at O are equal or supplementary, S_1, S_2, S_3, S_4 are proportional respectively to the rectangles under the containing sides; hence

dexter ratio of the question

$$= (OA.K + OC.K) : (OB.K + OD.K)$$

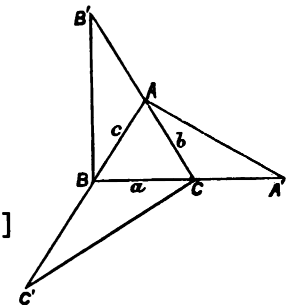
$$= AC : BD,$$

where $$K^2 = OA.OC = OB.OD.$$

11699. (Professor MOREL.)—Par les sommets A, B, C d'un triangle acutangle on mène les droites AA′, BB′, CC′ respectivement perpendiculaires aux côtés AB, BC, CA, et limitées aux côtés opposés aux sommets A, B, C. Démontrer que le triangle est équilatéral, si l'on a

$$\frac{AB}{AA'} + \frac{BC}{BB'} + \frac{CA}{CC'} = \sqrt{3}.$$

Solution by Professor DROZ FARNY; Rev. D. THOMAS, M.A.; *and others.*



$\frac{AB}{AA'} = \cot B$, $\frac{BC}{BB'} = \cot C$, $\frac{CA}{CC'} = \cot A$.

Il s'agit donc de demontrer que le triangle est équilateral si

$$\cot A + \cot B + \cot C = \cot\omega = \sqrt{3};$$

or $$\Sigma(\cot A - \cot B)^2$$

$$= 2\Sigma\cot^2 A - 2\Sigma\cot A\cot B$$

$$= 2\left[(\cot A + \cot B + \cot C)^2 - 3(\cot A\cot B + \cot A\cot C + \cot B\cot C)\right]$$

$$= 2\left[\cot^2\omega - 3\right];$$

$\cot w$ ne peut donc atteindre sa valeur minimale $\sqrt{3}$ que si $\cot A = \cot B = \cot C$ dans le théorème énoncé.

[*Otherwise*:—$AA' = 2\Delta/a$, $BB' = 2\Delta/b$, $CC' = 2\Delta/c$; hence

$$bc + ca + ab = 4\Delta\sqrt{3}, \quad 3\Sigma a^4 - 5\Sigma b^2c^2 + 2abc\,.\,\Sigma a = 0,$$

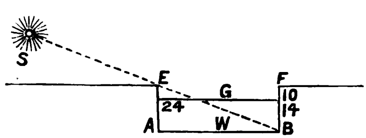
$$(\Sigma a^4 - \Sigma b^2c^2) - 2(\Sigma b^2c^2 - abc\,.\,\Sigma a) = 0, \quad \tfrac{3}{2}\Sigma(b^2-c^2)^2 - \Sigma a^2(b-c)^2 = 0;$$

therefore $$3\Sigma(b-c)^2\left\{(b+c)^2 - a^2\right\} + \Sigma a^2(b-c)^2 = 0,$$

and, because all the terms are positive, $a = b = c$.]

11092. (Professor SHIELDS, M.A.)—The water in a canal C is 10 feet below the top edges EE of the canal, and 14 feet deep. At a certain time of the day the sun's rays, or shadow S over the edge E of the canal, strikes the water W, 24 feet from the side wall A, in such a manner as to be in line of collimation, or strike the opposite lower edge B of the canal. Find the width of the canal.

Solution by Professor ZERR; Professor AIYAR; *and others.*



A cross-section of the canal being a rectangle, we have two similar right-triangles, with 10 and 24 for perpendiculars, and for bases 24 and x (= width of canal); hence $10 : 24 = 24 : x$,

or $$x = 57\cdot6 \text{ feet} = \text{width}.$$

6513. (Professor MINCHIN, M.A.)—A plane curve rolls without sliding with an angular velocity varying in any way, along a fixed plane curve; prove that the acceleration of the point of contact, considered not as a point in fixed space, but as a point of the rolling curve, is at any instant $\dfrac{\omega^2}{1/\rho \pm 1/\rho'}$, and show how to find the successive time rates of increase of the components of acceleration of this point parallel to the tangent and normal.

Solution by H. W. CURJEL, B.A.

Consider the motion during the infinitesimal interval of time dt, at the end of which the curves touch at B, and at the beginning of which A, A′ were in contact. The curves may evidently be replaced by their circles of curvature at A, A′.

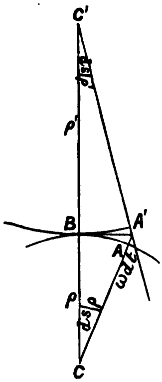


Fig. 1.

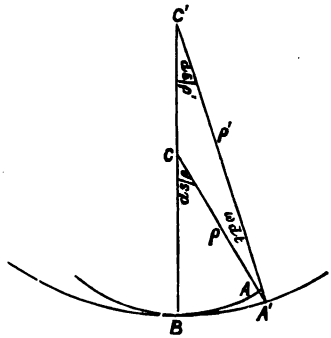


Fig. 2.

Let C, C′ be the centres of curvature, and x, x' the distances of A, A′ from the tangent at B.

Let ds = arc AB = arc A′B, since there is no sliding.

Then $$2x\rho = ds^2 = 2\rho' x', \quad AA' = x \pm x' = \frac{ds^2}{2\rho} \pm \frac{ds^2}{2\rho'},$$

also $$\frac{ds}{\rho} \pm \frac{ds}{\rho'} = \omega dt,$$

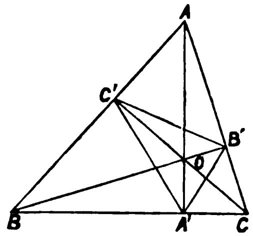
the upper sign being taken for external (Fig. 1), and the lower for internal (Fig. 2) contact; therefore

$$AA' = \frac{ds}{2}\,\omega\, dt = \tfrac{1}{2}\,\frac{\omega^2}{1/\rho \pm 1/\rho'}\,(dt)^2;$$

$$\therefore \text{ velocity} = \frac{\omega^2\, dt}{1/\rho \pm 1/\rho'}; \quad \therefore \text{ acceleration} = \frac{\omega^2}{1/\rho \pm 1/\rho'}.$$

11595. (EDITOR.)—If AOA_1, BOB_1, COC_1 are perpendiculars from the vertices of a triangle to the opposite sides, R the circum-radius, and Δ, Δ_1, Δ_a, Δ_b, Δ_c the areas of the triangles ABC, $A_1B_1C_1$, BOC, COA, AOB, prove that $\Delta\Delta_a\Delta_b\Delta_c = R^4\Delta_1^2$.

Solution by J. RICE; W. J. GREENSTREET, M.A.; *and others.*



Since Δ_1 is the pedal triangle of Δ, Δ_a, Δ_b, Δ_c, we have

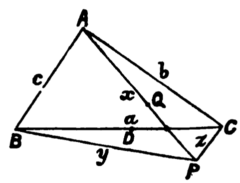
$$2\Delta = \Delta_1 \sec A \sec B \sec C,$$

$$2\Delta_a = \Delta_1 \sec A \operatorname{cosec} B \operatorname{cosec} C, \text{ \&c.};$$

$$\therefore \quad \Delta\Delta_a\Delta_b\Delta_c = 4\Delta_1^4 \frac{1}{64\Pi\sin^2 A \,.\, \Pi\cos^2 A} = \Delta_1^2\left(\frac{2\Delta_1}{\Pi\sin 2A}\right)^2 = R^4 \,.\, \Delta_1^2.$$

11523. (Professor BHATTACHARYA.)—If a, x; b, y; c, z be the lengths of the three pairs of opposite edges of a real tetrahedron, when will any two of the three (A) a^2+x^2, b^2+y^2, c^2+z^2, or (B) $(a+x)^2$, $(b+y)^2$, $(c+z)^2$, be together greater than the third?

Solution by H. W. CURJEL, B.A.



Let BC $= a$, CA $= b$, AB $= c$, and PA $= x$, PB $= y$, PC $= z$; where ABCP is the tetrahedron. Bisect BC in D and AP in Q; then

$$b^2+z^2 = 2\left\{\frac{x^2}{4}+CQ^2\right\},$$

$$c^2+y^2 = 2\left\{\frac{x^2}{4}+BQ^2\right\};$$

$$\therefore \quad b^2+y^2+c^2+z^2 = x^2+2(CQ^2+BQ^2)$$

$$= x^2+2\left\{2\left[\frac{a^2}{4}+QD^2\right]\right\} = x^2+a^2+4QD^2;$$

$$\therefore \quad (b^2+y^2)+(c^2+z^2) > x^2+a^2.$$

Again, if we rotate triangle BCP about BC till triangles ABC, BCP are in the same plane, we evidently increase AP, and therefore increase AP × BC. But when ABPC are all in a plane,

$$AP \times BC = \text{or} < AB \times PC + BP \times AC,$$

according as A, B, P, C are concyclic or not.

Therefore, when ABCP is a tetrahedron,

$$ax < by+cz; \quad \therefore \quad 2ax < 2by+2cz;$$

but $x^2+a^2 < b^2+y^2+c^2+z^2$; $\quad\therefore\quad (x+a)^2 < (b+y)^2+(c+z)^2$.

11433. (W. J. Greenstreet, M.A.)—If a, b, c are the sides of a triangle, and $\Sigma a^2/x = 0$, show that xyz is negative if $\Sigma(x)$ is positive.

Solution by H. W. Curjel, B.A.; *the* Proposer; *and others.*

$x+y+z > 0$; therefore the three numbers are not all negative, and $a^2/x+b^2/y+c^2/z = 0$; therefore they are not all positive.

Suppose y and z are negative, and are $-y'$, $-z'$. Then, a being $< b+c$, a^2 is $< b^2+c^2+2bc$. Also $x+y+z > 0$, $x > y'+z'$; therefore

$$\frac{a^2}{x} < \frac{b^2+c^2+2bc}{y'+z'}, \quad \text{and as} \quad \frac{a^2}{x} = \frac{b^2}{y'}+\frac{c^2}{z'}, \quad \frac{b^2}{y'}+\frac{c^2}{z'} < \frac{b^2+c^2+2bc}{y'+z'};$$

or $\quad (y'+z')(b^2z'+c^2y') < (b^2+c^2+2bc)\,y'z', \quad$ or $\quad (bz'-cy')^2 < 0$;

therefore one of the three numbers only is negative.

This theorem is the algebraical expression of the fact that, x, y, z being barycentric coordinates of a point with reference to the triangle ABC, the circumcircle lies in one of the ex-spaces of the triangle of reference.

7502. (W. G. Lax, B.A.)—If there be two parabolas in a plane, whose axes are in the same straight line and concavities turned in opposite directions towards one another, the vertices being at a distance h apart, and if one be of fixed latus rectum $4a$, but the other variable: find (1) the latus rectum of this latter when the area contained by the curve (variable) and the common chord of the two parabolas is a maximum; and (2) the position of the centre, and the area, of the circle inscribed to the figure formed by the two curves in this position.

Solution by H. J. Woodall, A.R.C.S.

Let $y^2 = 4ax$ and $y^2 = 4b(h-x)$ be the curves; these cut at

$$ax = b(h-x); \quad \text{therefore} \quad x = bh/(a+b) = x_1 \text{ say.}$$

Area of second curve to this ordinate

$$= \int_{x_1}^{h} y\,dx = \tfrac{4}{3}b^{\frac{1}{2}}\left\{ah/(a+b)\right\}^{\frac{3}{2}} = \tfrac{4}{3}u^{\frac{1}{2}}, \text{ say;}$$

therefore $\qquad u = ba^3h^3/(a+b)^3,$

$$\frac{du}{db} = a^3h^3\left\{(a+b)-3b\right\}/(a+b)^4 = 0, \quad \text{when} \quad b = \tfrac{1}{2}a;$$

therefore $y^2 = 4ax$ and $y^2 = 2a(h-x)$ are the curves.

(2) Normal at (xy) to $y^2 = 4ax$ $\qquad$ cuts axis at $(x+2a,\ 0)$,

,, $\qquad (x'y')$ to $y^2 = 2a(h-x)$ $\qquad$,, $\qquad (x'-a,\ 0)$,

and these normals meet on the axis; therefore $x' = x+3a$. But, because the normals are equal, $\qquad y^2+4a^2 = y'^2+a^2$;

therefore $\quad 4ax+4a^2 = 2a(h-x')+a^2 = 2ah+a^2-2a(x+3a)$;
therefore $\quad 6ax = 2ah-9a^2$; therefore $x = \frac{1}{6}(2h-9a)$;
therefore radius $= (y^2+4a^2)^{\frac{1}{2}} = (4a^2+4ax)^{\frac{1}{2}}$

$$= 2\left\{a^2+\tfrac{1}{6}(2ah-9a^2)\right\}^{\frac{1}{2}} = 2\left(\tfrac{1}{3}ah-\tfrac{1}{2}a^2\right)^{\frac{1}{2}}.$$

11698. (Professor Barisien.)—D'un point P du plan d'une ellipse, on abaisse les quatre normales à l'ellipse dont les pieds sont A, B, C, D. Montrer que le lieu des points, tels que le foyer F et le symétrique P′ de P par rapport au centre, soient sur une même conique que les pieds A, B, C, D, est une ellipse (E). Cette ellipse (E) est semblable à l'ellipse donnée; elle a son centre au second foyer F′ de l'ellipse donnée, et elle passe par les sommets du petit axe de cette ellipse.

Solution by Prof. Droz Farny; R. F. Davis, M.A.; *and others.*

Soient $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

l'équation de l'ellipse, et α, β les coordonnées du point P.

Toute conique passant par les 4 points A, B, C, D aura une équation de la forme $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\lambda\left[a^2\alpha y-b^2\beta x-c^2xy\right]=0.$$

Si les points $(c, 0)$ et $(-\alpha, -\beta)$ doivent se trouver sur cette conique, on aura $$\lambda a^2\beta c = 1, \quad \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1+2\lambda c^2\alpha\beta = 0 \quad \text{..............(1, 2).}$$

En éliminant λ entre ces 2 équations et en remplaçant α, β par x et y, on trouve le lieu $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2cx}{a^2}=1,$$

ellipse semblable à l'ellipse proposée. L'équation est vérifiée pour
$$x=0, \quad y=\pm b.$$

En prenant les dérivées partielles par rapport à x et à y, on obtient pour les coordonnées du centre $x+c=0$ ou $x=-c$ et $y=0$.

11608. (R. F. Davis, M.A.)—If 1, ω, ω^2 be the cube roots of unity,
$A \equiv yz+\omega zx+\omega^2xy$, $B \equiv x+\omega y+\omega^2z$, $C \equiv yz+\omega^2zx+\omega xy$, $D \equiv x+\omega^2y+\omega z$,
$$S \equiv (y-z)(z-x)(x-y),$$
(1) prove by substitution that the equation $A/B = C/D$ is satisfied by either $y=z$, or $z=x$, or $x=y$; (2) verify what is thereby suggested, that $AD-BC$ can be thrown into the form λS, where $\lambda = \omega-\omega^2 = \sqrt{(-3)}$; (3) exhibit $4BD \cdot AC$, and $(AD+BC)^2$ as rational functions of xyz, whence also $3S^2$; (4) deduce the criterion that $t^3+tp^2+qt+r=0$ should have two equal roots.

Solution by H. J. WOODALL, A.R.C.S.

(1) $x = y$ gives $A = x(z + \omega z + \omega^2 x)$, $B = x + \omega x + \omega^2 z$,
$C = x(z + \omega^2 z + \omega x) = \omega^2 A$, $D = x + \omega^2 x + \omega z = \omega^2 B$;
therefore $A/B = C/D$. Similarly, if $y = z$ or $z = x$,

(2)
$$AD = 3xyz + \omega^2 (y^2 z + z^2 x + x^2 y) + \omega (yz^2 + zx^2 + xy^2),$$
$$BC = 3xyz + \omega (y^2 z + z^2 x + x^2 y) + \omega^2 (yz^2 + zx^2 + xy^2);$$
$$\therefore\ (AD - BC) = (\omega^2 - \omega) \left\{ (y^2 z + z^2 x + x^2 y) - (yz^2 + zx^2 + xy^2) \right\}$$
$$= (\omega - \omega^2)(y-z)(z-x)(x-y) = (\omega - \omega^2)\, S = (-3)^{\frac{1}{2}}\, S;$$
therefore $(AD - BC)^2 = -3S^2$, $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$.

(3)
$$AD \,.\, BC = 9x^2y^2z^2 + 3xyz(\omega + \omega^2)(x+y)(y+z)(z+x)$$
$$+ (y^2 z + z^2 x + x^2 y)^2 \omega^3 + \omega^3 (yz^2 + zx^2 + xy^2)^2$$
$$+ (\omega^2 + \omega^4)(y^2 z + z^2 x + x^2 y)(yz^2 + zx^2 + xy^2)$$
$$= 9x^2y^2z^2 - 3xyz(x+y)(y+z)(z+x) + (y^2 z + z^2 x + x^2 y)^2$$
$$+ (yz^2 + zx^2 + xy^2)^2 - (y^2 z + z^2 x + x^2 y)(yz^2 + zx^2 + xy^2)$$
$$= \left\{ 3xyz - (y^2 z + z^2 x + x^2 y) \right\} \left\{ 3xyz - (yz^2 + zx^2 + xy^2) \right\} + (y-z)^2 (z-x)^2 (x-y)^2;$$
therefore
$$(AD + BC)^2 = (AD - BC)^2 + 4AD \,.\, BC$$
$$= 4 \left\{ 3xyz - (y^2 z + z^2 x + x^2 y) \right\} \left\{ 3xyz - (yz^2 + zx^2 + xy^2) \right\}$$
$$+ (y-z)^2 (z-x)^2 (x-y)^2.$$

[The PROPOSER adds the following developments :—

From the values of AD and BC, given above in (2), we have, immediately,
$$AD + BC = 6xyz - (y^2 z + yz^2 + \ldots)$$
$$= 8xyz - (y+z)(z+x)(x+y),$$
or
$$= 9xyz - (x+y+z)(yz + zx + xy).$$

Also, by simple multiplication,
$$BD = x^2 + y^2 + z^2 - yz - zx - xy$$
$$= (x+y+z)^2 - 3(yz + zx + xy)^2,$$
$$AC = y^2z^2 + z^2x^2 + x^2y^2 - xyz(x+y+z)$$
$$= (yz + zx + xy)^2 - 3xyz(x+y+z).$$

Substituting in the identity $-(AD - BC)^2 = 4BD \,.\, AC - (AB + CD)^2$, we have
$$3S^2 = 4\left\{ yz + zx + xy - x^2 - y^2 - z^2 \right\} \left\{ xyz(x+y+z) - y^2z^2 - z^2x^2 - x^2y^2 \right\}$$
$$- \left\{ (y+z)(z+x)(x+y) - 8xyz \right\}^2,$$
the form given in 11642, which is Question V. in the Three-day Problem Paper of the Cambridge Mathematical Tripos for 1892.

If x, y, z be the roots of the equation $t^3 + pt^2 + qt + r = 0$; then we have
$$3S^2 = 4(p^2 - 3q)(q^2 - 3pr) - (9r - pq)^2,$$
and S vanishes when the equation has equal roots.]

11291. (Professor ORCHARD, M.A., B.Sc.)—A uniform right cone, floating vertex downward, sinks so as to be just immersed before rising, when a weight (= the cone's weight) is placed upon the base; find the volume immersed when floating freely.

Solution by H. W. CURJEL, B.A.

Let h = the height of the cone, and x = the part of the axis immersed when the cone is floating freely, and kh^3 = volume of cone, and ρ is density. Then volume immersed = $kx^3 = \rho kh^3$. The work done by gravity and the pressure of the water until the cone is totally immersed = 0; hence

$$\int_x^h (2kh^3\rho - kx^3)\, g\, dx = 0; \qquad \therefore \quad 2h^3\rho\,(h-x) - \tfrac{1}{4}(h^4 - x^4) = 0.$$

But $\rho = x^3/h^3$; hence x is the positive root of $7x^3 - hx^2 - h^2x - h^3 = 0$.

This determines the volume immersed, which $= kx^3$.

11644. (J. W. RUSSELL, M.A.)—Of the lines joining corresponding pairs of points of two homographic ranges on a conic, two pass through any given point.

Solution by the PROPOSER.

Let O be the given point, and let (ABC ...) and (A′B′C′ ...) be the homographic ranges on the conic. Let OA cut the conic again in A″, OB in B″, and so on. Then, by involution, (A″B″C″ ...) = (ABC ...) = (A′B′C′ ...) by hypothesis. Hence (A′B′C′ ...) and (A″B″C″ ...) are homographic ranges on the conic. The required rays are the lines joining O to the common points of these two homographic ranges.

11306. (A. J. PRESSLAND.)—Prove that the polar of the centroid of a triangle with respect to any escribed parabola is a tangent to the minimum ellipse.

Solution by H. W. CURJEL, B.A.

Project the triangle orthogonally into an equilateral triangle. The parabola will remain a parabola, and the centroid of the triangle will project into the centroid of the equilateral triangle, and will therefore coincide with its orthocentre and circumcentre. The minimum circum-ellipse becomes the circumcircle. Since the centroid O is the orthocentre of triangle ABC, O lies on the directrix, and circle ABC passes through the focus S of the parabola; therefore the polar of O is perpendicular to OS through S, and therefore touches the circle at S. Hence in the original figure we get the required result.

5301. (Rev. E. Hill, M.A.)—Certain persons have imagined the existence of a subterranean connexion between the waters of the Dead Sea and the Mediterranean. Although the difference of their levels is 1300 feet, yet, since the ratio of their densities is 1·24, it is possible that such a passage may exist. But find its necessary depth.

Solution by H. W. Curjel, B.A.

Let x = necessary depth of passage below the Dead Sea; then

$$x \times 1{\cdot}24 = x + 1300; \quad \therefore \; x = 1300/{\cdot}24 \text{ feet} = 5416 \text{ feet } 8 \text{ inches}.$$

Or depth = 6716 feet 8 inches below the Mediterranean.

[Professor Zerr solves the problem thus :— Since 1 foot of Dead Sea water = 24 feet Mediterranean, the necessary depth is $1300/23 = 56\frac{12}{23}$ feet below the surface of the Dead Sea, or $1356\frac{12}{23}$ feet below the surface of the Mediterranean.]

11555. (Professor de Longchamps.)—On considère une hyperbole équilatère H ; par l'un des foyers F on mène Δ parallèle à l'une des asymptotes de H. D'un point M, mobile sur H, on abaisse une perpendiculaire MP sur Δ. Démontrer que le cercle inscrit au triangle FMP a un rayon invariable.

Solution by R. Knowles; Professor Sarkar; *and others.*

The equations to H, Δ are $x^2 - y^2 = a^2$, $x - y = 2^{\frac{1}{2}}a$; the coordinates of M, F, P are respectively x_1, y_1; $2^{\frac{1}{2}}a, 0$; $\frac{1}{2}(x_1 + y_1 + 2^{\frac{1}{2}}a)$, $\frac{1}{2}(x_1 + y_1 - 2^{\frac{1}{2}}a)$; whence we find

$$\text{FP} = (x_1 + y_1 - 2^{\frac{1}{2}}a)/2^{\frac{1}{2}}, \quad \text{PM} = (x_1 - y_1 - 2^{\frac{1}{2}}a); \quad \text{FM} = a - 2x_1.$$

In the triangle FMP, the angle at P is a right angle;

$\therefore$ radius of inscribed circle $= \text{FP}\,.\,\text{PM}/(\text{FP} + \text{PM} + \text{FM}) = \frac{1}{2}(a - 2^{\frac{3}{2}})$, and is invariable.

10966. (Professor Déprez.) — On considère tous les triangles sphériques ABC, inscrits à un même petit cercle, ayant un sommet fixe A et dans lesquels la somme $\cos AB + \cos AC$ a une valeur constante. Démontrer que (1) le point de rencontre des médianes décrit un grand cercle; (2) la base BC enveloppe une ellipse sphérique.

Solution by W. J. Greenstreet, M.A.

1. $\text{Cos}\,AB + \cos AC = 2 - 2(\sin^2 \frac{1}{2}AB + \sin^2 \frac{1}{2}AC) = 2 - 1/2R^2\,.\,(b^2 + c^2)$, where the chord AB = c, and chord AC = b.

If the median AM meet the circle in N,

$$b^2 + c^2 = 2(\text{AM}^2 + \text{BM}^2) \quad \text{and} \quad \text{AM}(\text{AN} - \text{AM}) = \text{BM}^2;$$

$$\therefore \; \text{AM}\,.\,\text{AN} = \text{AM}^2 + \text{BM}^2 = \tfrac{1}{2}(b^2 + c^2) = \text{constant},$$

i.e., M, N are inverses with respect to the centre A.

Therefore the locus of M and of the centre of gravity of the triangle are two lines perpendicular to the diameter AOD of the sphere.

As the medians of the spherical triangle meet at the extremities of a diameter of the sphere, the locus is the great circle through A.

2. If SM meet the arc BC in M′, then M′ is the projection of the pole P of the circle ABC on the arc. The locus of M′ is a great circle X passing through the locus of M.

If p_1, p_2 are the "spheric symmetrics" of p for two positions of the arc BC and of one of the points of intersection of these two positions, then the locus of p_1, p_2 is a small circle x parallel to X.

The spherical triangle $p_1 q p_2$ is isosceles, so that the contact point T of the arc BC and its envelope is on the great circle perpendicular to that which touches x in p_1.

If Q is the pole of X on the same side of the sphere as P, then Q is the pole of x, and $Qp_1 = QT + Tp = QT + TP =$ constant;
i.e., BC envelops two ellipses, foci P, Q and P′, Q′, respectively, where P′, Q′ are diametrically opposite P and Q.

11672. (Morgan Brierley.)—Given the base, the vertical angle, and the sum of the squares of the lines drawn from the vertical angle to bisect the segments of the base made by the foot of the perpendicular from the same point.

Solution by R. F. Davis, M.A.

Let BC be the given base, and O its mid-point; then the vertex P of the required triangle lies on a fixed circle passing through B, C.

Draw PN perpendicular to BC; and let b, c be the mid-points of BN, CN. If PQ = NP, the locus of Q is a determinate circle of twice the linear dimensions of the circle PBC. The position of Q is determined on this circle, since OQ is known; for

$$2\,OQ^2 + 2\,OB^2 = QB^2 + QC^2 = 4\,(Pb^2 + Pc^2) = \text{known}.$$

11684. (Professor Lampe, LL.D.) — When will the number of a Sunday after Trinity coincide with the number which indicates its date (day of the month)? [Suggested by the following Question 4288:— "The 19th Sunday after Trinity fell this year (1873) on the 19th of October; when will this coincidence recur?"]

Solution by the Proposer.

The date of a fixed Sunday after Trinity depends solely upon the date of Easter, and will fall on the same day of the year, whenever the date of Easter Sunday is repeated. Hence it suffices to establish a table of the Sundays after Trinity for all positions of Easter from March 22nd to April 25th. The subjoined table furnishes the following coincidences:—

TABLE I.

Number of Sunday after Trinity.	Date.	Easter.
1	June 1	Mar. 30
2	June 2	,, 24
	July 2	Apr. 23
3	,, 3	,, 17
4	,, 4	,, 11
5	,, 5	,, 5
6	,, 6	Mar. 30
7	,, 7	,, 24
	Aug. 7	Apr. 24
8	,, 8	,, 18
9	,, 9	,, 12
10	,, 10	,, 6
11	,, 11	Mar. 31
12	,, 12	,, 25
	Sept. 12	Apr. 25
13	,, 13	,, 19
14	,, 14	,, 13
15	,, 15	,, 7
16	,, 16	,, 1
17	,, 17	Mar. 26
	Oct. 17	Apr. 25
18	,, 18	,, 19
19	,, 19	,, 13
20	,, 20	,, 7
21	,, 21	,, 1
22	,, 22	Mar. 26
23	Nov. 23	Apr. 20
24	,, 24	,, 14
25	,, 25	,, 8
26	,, 26	,, 2

TABLE II.

Day	Month of Easter.	Date of Coincidence.
22	March	0
23	,,	0
24	,,	June 2, July 7
25	,,	Aug. 12
26	,,	Sept. 17, Oct. 22
27	,,	0
28	,,	0
29	,,	0
30	,,	June 1, July 6
31	,,	Aug. 11
1	April	Sept. 16, Oct. 21
2	,,	Nov. 26
3	,,	0
4	,,	0
5	,,	July 5
6	,,	Aug. 10
7	,,	Sept. 15, Oct. 20
8	,,	Nov. 25
9	,,	0
10	,,	0
11	,,	July 4
12	,,	Aug. 9
13	,,	Sept. 14, Oct. 19
14	,,	Nov. 24
15	,,	0
16	,,	0
17	,,	July 3
18	,,	Aug. 8
19	,,	Sept. 13, Oct. 18
20	,,	Nov. 23
21	,,	0
22	,,	0
23	,,	July 2
24	,,	Aug. 7
25	,,	Sept. 12, Oct. 17

A table of the dates of Easter will then show in which years a coincidence of this kind will happen. Question 4288 asks for the recurrence of the coincidence of the 19th Sunday after Trinity with the 19th of October (in which year the 14th Sunday after Trinity will at the same time fall on the 14th of September). We add, therefore, from a table of Easter, all years, from 1582 to 2200, with Easter on the 13th of April:—

1653, 1659, 1664, 1721, 1727, 1732, 1800, 1873, 1879, 1884,
1941, 1952, 2031, 2036, 2104, 2183, 2188.

The second table, arranged after the date of Easter, shows that there are 13 dates of Easter, out of 35, which do not admit of a similar coincidence.

If Easter falls on April 5th, the fifth Sunday after Trinity will be July 5th. This is the only case when the date of Easter agrees with the number of Sunday after Trinity which coincides with its date. This singular coincidence will occur in 1896, 1931, 1942, 1953, &c.

7162. (H. L. Orchard, M.A.)—ABC is a "perfectly rough" inclined plane. When AC is base a sphere rolls down in the same time that a cylinder does when AB is base. Find the angle of the plane.

Solution by H. W. Curjel, B.A.

Let $\angle BCA = \alpha$, then $\angle CBA = \frac{1}{2}\pi - \alpha$. Then, using the usual notation, we have, for the motion of the sphere,

$$ma\ddot{x} + mk^2\ddot{\theta} = mga\sin\alpha\,; \quad \text{and} \quad x = a\theta\,; \quad \therefore\ \ddot{x} = a\ddot{\theta}\,;$$

$$\therefore\ \ddot{x} = \frac{a^2}{a^2+k^2}\,g\sin\alpha = \tfrac{5}{7}g\sin\alpha, \quad \text{since} \quad k^2 = \tfrac{2}{5}a^2\,;$$

$$\therefore\ (\text{time down BC})^2 = 2\text{BC}/(\tfrac{5}{7}g\sin\alpha).$$

Similarly, for the cylinder,

$$\ddot{x} = \frac{a^2}{a^2+k^2}\,g\cos\alpha = \tfrac{2}{3}g\cos\alpha, \quad \text{since} \quad k^2 = \tfrac{1}{2}a^2\,;$$

$$\therefore\ (\text{time down CB})^2 = 2\text{BC}/(\tfrac{2}{3}g\cos\alpha)\,; \quad \therefore\ \tfrac{2}{3}\cos\alpha = \tfrac{5}{7}\sin\alpha\,;$$

$$\therefore\ \alpha = \tan^{-1}\tfrac{14}{15} = 43^\circ\,1'\,30'' \text{ nearly}.$$

11063. (Artemas Martin, LL.D.)—Five bricks are placed upon one another (in the form of a wall) at random: find the probability that the pile will fall down.

Solution by H. W. Curjel, B.A.

If all the bricks are equal in length, the chance that the top one will not fall if the second one is held firm is evidently $\frac{1}{2}$, for total range $= 2 \times$ length of brick, and with condition for steadiness, range $= l$, where $l =$ length of a brick. Similarly, if the top brick is steady when the second one is held firm, the chance that the two top ones will not fall when the third is held $= \frac{1}{2}$, and so on. Hence, chance that the pile will stand $= (\frac{1}{2})^4$; therefore chance the pile will fall $= 1 - (\frac{1}{2})^4$.

11657. (Professor DE WACHTER.)—Pour que les équations
$$y^2+z^2-2ayz=0,\quad z^2+x^2-2bzx=0,\quad x^2+y^2-2cxy=0$$
soient compatibles, il faut et il suffit que l'on ait $a^2+b^2+c^2-2abc=1$.

Solution by H. W. CURJEL, B.A.; R. F. DAVIS, M.A.; *and others.*

The equations may be written
$$\tfrac{1}{2}(y/z+z/y)=a,\quad \tfrac{1}{2}(z/x+x/z)=b,\quad \tfrac{1}{2}(x/y+y/x)=c.$$
Hence, if the equations are consistent, we have
$$a^2+b^2+c^2-2abc=\tfrac{1}{4}\Sigma(y^2/z^2+z^2/y^2+2)-\tfrac{2}{8}(y/z+z/y)(y/x+x/y)(x/z+z/x)$$
$$=\tfrac{1}{4}\Sigma(y^2/z^2+z^2/y^2)+\tfrac{6}{4}-\tfrac{1}{4}\left[2+\Sigma(y^2/z^2+z^2/y^2)\right]=1.$$
Also, if x, y, z satisfy the first two equations, and $a^2+b^2+c^2-2abc=1$,
then $a^2+b^2+c^2-2abc=1=\tfrac{1}{4}\Sigma(y/z+z/x)^2-\tfrac{2}{8}(y/z+z/y)(z/x+x/z)(x/y+y/x)$
$$=a^2+b^2+\tfrac{1}{4}(x/y+y/x)^2-ab(x/y+y/x);$$
therefore $\qquad c^2-2abc=\tfrac{1}{4}(x/y+y/x)^2-ab(x/y+y/x),$
$$\tfrac{1}{2}(x/y+y/x)=c\quad\text{or}\quad 2ab-c,$$
one value being given by $\qquad \dfrac{x}{y}=\dfrac{1+(a^2-1)^{\frac{1}{2}}}{1+(\quad-1)^{\frac{1}{2}}},$

and the other by $\qquad \dfrac{x}{y}=\dfrac{1+(a^2-1)^{\frac{1}{2}}}{1-(b^2-1)^{\frac{1}{2}}}.$

Hence the condition is sufficient.

11673. (H. J. WOODALL, A.R.C.S.) — Place 4 sovereigns and 4 shillings in close alternate order in a line. Required in four moves, each of two contiguous pieces (without altering the relative positions of the two), to form a *continuous* line of 4 sovereigns followed by 4 shillings.

Solution by R. F. DAVIS, M.A.

Let A*a*B*b*C*c*D*d* be the initial order, the capitals denoting sovereigns. Then successive moves give

A*a*B*c*D*db*C, AB*ca*D*db*C, ABD*db*C*ca*, ABDC*cadb*.

[The PROPOSER asserts that this "Solution is wrong."]

11618. (Professor RAMASWAMI AIYAR, M.A.)—Let ABC be a triangle inscribed in a parabola; through its incentre or an ex-centre let the diameter of the parabola be drawn, meeting the parabola in P. Then the tangent at P to the parabola is also a tangent to the circumcircle of ABC. [From this we may obtain a method for drawing common tangents to a circle and a parabola intersecting in four points.]

Solution by R. F. DAVIS, M.A.

Let ABC be a triangle inscribed in a parabola, I its in-centre. Then, if p_1, p_2, p_3 be the perpendiculars from A, B, C on the diameter of the parabola passing through I, $ap_1 + bp_2 + cp_3 = 0$.

[For I may be regarded as the centroid of masses a, b, c at A, B, C respectively, so that the above is true for *any* line passing through I.]

If q_1, q_2, q_3 be the perpendiculars from A, B, C upon the tangent to the parabola at the extremity of the above diameter, then, from the nature of the curve, $p_1^2 : p_2^2 : p_3^2 = q_1 : q_2 : q_3$;

hence $$a\sqrt{q_1} + b\sqrt{q_2} + c\sqrt{q_3} = 0;$$

which requires the tangent to the parabola to be also a tangent to the circumcircle of ABC.

11642. (R. F. DAVIS, M.A.)—If

$$P = yz + zx + xy - x^2 - y^2 - z^2, \quad Q = (y+z)(z+x)(x+y) - 8xyz,$$
$$R = xyz(x+y+z) - y^2z^2 - z^2x^2 - x^2y^2;$$

prove that $$4PR - Q^2 = 3(y-z)^2(z-x)^2(x-y)^2.$$

Solution by H. J. WOODALL, A.R.C.S.; Professor ZERR; *and others.*

$$-2P = (y-z)^2 + (z-x)^2 + (x-y)^2, \quad Q = x(y-z)^2 + y(z-x)^2 + z^2(x-y)^2,$$
$$-2R = x^2(y-z)^2 + y^2(z-x)^2 + z^2(x-y)^2;$$
$$\therefore \quad 4PR - Q^2 = \Sigma (y-z)^2(z-x)^2\{x^2 - 2xy + y^2\} = \Sigma (y-z)^2(z-x)^2(x-y)^2$$
$$= 3(y-z)^2(z-x)^2(x-y)^2.$$

7306. (Professor HUDSON, M.A.)—From a point P on a parabola, focus S, PM, PN are drawn perpendicular to the directrix and axis; PT, PG are the tangent and normal limited by the axis. What line represents the resultant of forces represented by PM, PT, PS, PN, PG?

Solution by Professor SCHOUTE.

If $y^2 = 2px$ is the equation of the parabola, we find for the X and Y components of the forces in the indicated order:—

$$\begin{array}{llllll} X & \ldots & -(x+\tfrac{1}{2}p), & -2x, & -(x-\tfrac{1}{2}p), & 0, \quad p, \\ Y & \ldots & 0, & -y, & -y, & -y, \quad -y, \end{array}$$

or, of the resultant, $X = -4x + p$, $Y = -4y$.

This proves that the resultant is found by joining $P(x, y)$ to the mid-point U of the axial segment limited by vertex and focus, and taking $PV = 4PU$ on this line. [A solution identical with the PROPOSER's own will be found on p. 59 of Vol. LVIII.]

11529. (Professor MALET, F.R.S.)—The two quadrics U and U + LM intersect in the planes L and M. If A be a point on U, and B a point on U + LM, such that the tangent planes at A and B intersect on L, and if the line AB cut the quadrics U and U + LM again in C and D respectively, prove that the tangent planes at C and D intersect on L, and that the tangent planes at A and D intersect on M, as do also the tangent planes at B and C.

Solution by the PROPOSER.

Consider two spheres, and take two points A and B, one on each sphere, such that the tangent plane at A is parallel to that at B, and let the line AB cut the spheres again respectively in C and D. Now, O and O_1 being the respective centres of the spheres, since AO is parallel to BO_1, the points O, O_1, A, B, C, D lie in the same plane. Now, since the tangent line at A to the circle through A and C with centre O is parallel to the tangent line through B to the circle through B and D with centre O_1, therefore the tangent at C is parallel to that at D, and the tangents at A and D intersect on the radical axis of the circles, as do also the tangents at B and C. Now the tangent planes to the spheres at A, B, C and D being perpendiculars to the plane of section through the corresponding tangent lines to the circles, and the finite plane of intersection of the spheres being a perpendicular to the same plane through the radical axis of the circles, the theorem in the question is true for spheres, and, being projective, is therefore true for any pairs of quadrics intersecting in planes.

11690. (Professor DE WACHTER.)—Démontrer que la somme des septièmes et des cinquièmes puissances des *n* premiers nombres entiers est égale au double du carré de la somme des cubes de ces mêmes nombres.

Solution by Rev. Dr. BRUCE; H. W. CURJEL, B.A.; *and others.*

By applying method for summation of series, we have

$$\Sigma n^7 = \&\text{c.}; \qquad \Sigma n^5 = \tfrac{1}{12}(2n^6+6n^5+5n^4-n^2);$$

$$\Sigma n^5+\Sigma n^7 = \tfrac{1}{8}n^4(n^4+4n^3+6n^2+4n+1) = 2\left[\tfrac{1}{4}n^2(n+1)^2\right]^2 = 2(\Sigma n^3)^2.$$

11705. (HERBERT ORFEUR.)—Show that (1), the first day of the year

$$\left(4p+\begin{matrix}0\\1\\2\\\text{or }3\end{matrix}\right)100+4n+a \text{ comes on the } \left(5n+a+\begin{matrix}\text{1st}\\\text{6th}\\\text{4th}\\\text{or 2nd}\end{matrix}\right) \text{ day of the week,}$$

where p, n, and a are integers, and $a > 0$ and < 5; and (2), for other years,

$$\text{the first day of the year comes on the } \left(\begin{matrix}\text{7th}\\\text{6th}\\\text{4th}\\\text{or 2nd}\end{matrix}\right) \text{ day of the week.}$$

Solution by the Rev. D. Thomas, M.A.

The question may be stated thus:—Show that the first day of the year $(4p+k)\,100+4n+a$ falls on the $(5n+5k+a+1)^{\text{th}}$ day of the week in common years, and on $(5n+5k)^{\text{th}}$ day in leap years.

Sunday letter is found from $2\,(\tfrac{1}{4}c)_r+2\,(\tfrac{1}{4}y)_r+4\,(\tfrac{1}{7}y)_r+1$;

$\therefore$ Sunday letter is $2k+2a+2n+4a+1$;

$\therefore$ first day of the year falls on the $9-(2k+2n+6a+1)^{\text{th}}$ day of the week,

i.e., on $(5n+5k+a+1)^{\text{th}}$ day of the week.

In leap years, because the Sunday letter of January is one more than that found above, the first day of the year falls a day earlier in each case.

11693. (Professor Orchard, M.A., B.Sc.)—Prove that

$$x^n = (x-3)(x-3^2)\dots(x-3^n)+3\,(x-3^2)(x-3^3)\dots(x-3^n)$$
$$+3^2x\,(x-3^3)\dots(x-3^n)+3^3x^2\,(x-3^4)\dots(x-3^n)+\dots+3^n\,.\,x^{n-1}.$$

Solution by T. Savage; H. J. Woodall; *and others.*

$$(x-3)(x-3^2)\dots(x-3^n)$$
$$=x^n-x^{n-1}\,.\,\Sigma 3^n+x^{n-2}\,(3\,.\,3^2+3\,.\,3^3+\dots+3^2\,.\,3^3+\dots+3^{n-1}3^n)+\dots$$
$$\dots+(-1)^{n-1}\,.\,3^n\,!\left(\frac{1}{3}+\frac{1}{3^2}+\dots\frac{1}{3^n}\right)x+(-1)^n\,.\,3\,.\,3^2.3^3\dots3^n,$$

$$3\,(x-3^2)\dots(x-3^n)=3\,.\,x^{n-1}-3\,(3^2+3^3+\dots3^n)\,x^{n-2}+\dots$$
$$\dots+(-1)^{n-2}\,.\,3^n!\left(\frac{1}{3^2}+\dots\frac{1}{3^n}\right)x+(-1)^{n-1}\,.\,3\,.\,3^2.\,3^3\dots3^n,$$

$$3^2x\,(x-3^3)\dots(x-3^n)$$
$$=3^2.\,x^{n-1}-3^2\,(3^3+3^4+\dots3^n)\,x^{n-2}+\dots+(-1)^n.\,x\,.\,\frac{3^n!}{3},$$

$$3^3\,.\,x^2\,(x-3^4)\dots(x-3^n)$$
$$=3^3\,.\,x^{n-1}-3^3\,(3^4+\dots3^n)\,x^{n-2}+\dots+(-1)^{n-1}\,.\,x^2\,.\,\frac{3^n!}{3\,.\,3^2},$$

$$\dots\quad\dots\quad\dots\quad\dots\quad\dots$$

$$3^n\,.\,x^{n-1}=3^n\,.\,x^{n-1};$$

therefore sum of left-hand expressions $=x^n$.

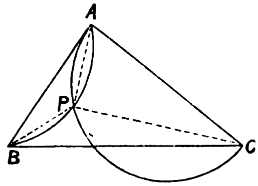
11271. (Editor.)—Construct a triangle having given the incentre I, the middle point of the base BC, and the foot of the perpendicular from A on BC.

Solution by Professor Ramaswami Aiyar.

A solution to this problem was given by Mr. Biddle on p. 115 of Vol. LVI.; but an immediate and better solution may be obtained from the property that the line joining the middle point of BC to I meets the perpendicular from A at a distance from A equal to r the in-radius.

11773. (H. J. Woodall, A.R.C.S.)—Find the locus of the intersection of two equal circles which are described on two sides AB, AC of a triangle as chord.

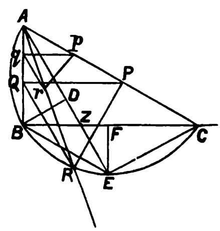
Solution by J. C. St. Clair; Professor Zerr; *and others.*



Let the circles intersect in P. Then the angles PBA, PCA are either equal or supplementary, and if we take different positions of P, the pencils (B . APP′...) and (C . APP′...) are inversely similar. Therefore P lies on a rectangular hyperbola passing through ABC. If we make P coincide with B and C, it is evident that the tangents at those points make equal alternate angles ($B \sim C$) with BC, which is therefore a diameter. If AB = AC, the curve reduces to the line BC, and the perpendicular upon it through A.

11503. (W. J. Greenstreet, M.A.)—In a right-angled triangle ABC, draw Bl perpendicular to the hypotenuse AC, lm perpendicular to AB, mn perpendicular to AC, np perpendicular to AB, and so on. Find the sum of these perpendiculars in terms of the sides of the triangle. From a point P in AC, a perpendicular PQ is let fall on AB; if $PQ^2 = AP . PC$, find P. Draw BD and CE perpendicular to the bisector AZ of the angle A; show that the middle point of BC, B, D, E are concyclic, and that the area of the triangle BDE is equal to BD . AE.

Solution by H. J. Woodall, A.R.C.S.



(1) $Bl = ac/b$, $lm = ac^2/b^2$, and so on; therefore sum

$$= ac/b \left\{1 + c/b + c^2/b^2 + \dots \infty\right\} = ac/(b-c).$$

(2) Take any point p in AC, draw pq perpendicular to AB, pr perpendicular to AC, make $pr = pq$, join Ar, qr; therefore Ar is locus of points r such that $pr = rq$. Next on AC, as diameter, describe circle ABC; this cuts Ar at R. Draw the triangle PQR with sides parallel to sides of pqr. Then PQ is the required straight line. Because angle CRA is a right angle, therefore $RP^2 = CP . PA$: also, PQR is similar to pqr, therefore PQ = PR; therefore $PQ^2 = AP . PC$ as required.

(3) Because AZ bisects angle A, therefore Az bisects arc BC; therefore E, the foot of the perpendicular from C on Az, is the mid-point

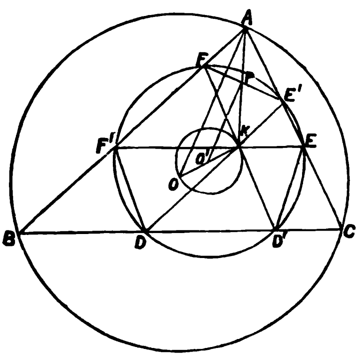
of the arc. If F be mid-point of BC, EF is perpendicular to BC; also BDE is a right angle; therefore B, D, F, E are concyclic, the circle being that on BE as diameter.

(4) $\triangle$BDE is similar and equal to $\triangle$BFE, and BD = EF; therefore

$$\triangle BDE = \tfrac{1}{4} \, . \, BD \, . \, BC.$$

11638. (H. Brocard.)—Démontrer que le cercle de Brocard et le premier cercle de Lemoine sont concentriques.

Solution by J. Dale.



Let ABC be any triangle, O the circumcentre, K the symmedian point, EF′, D′F, E′D respectively parallel to BC, CA, AB: then the points D, D′, E, E′, F, F′ are concyclic, and the circle passing through them is "le premier cercle de Lemoine."

AFKE′ is a parallelogram; $\therefore$ E′F is anti-parallel to BC, and $\angle$ AFE′ = $\angle$ ACB; so also $\angle$ BF′D = $\angle$ ACB; therefore E′F = DF′; and similarly also D′E is equal to E′F, or DF′.

Join AO, and AO is perpendicular E′F, and the perpendicnlar from O′ (the middle point of OK) passes through the middle point of E′F, and is equal to ½OA; similarly also the perpendiculars from O′ on D′E and DF′ pass through their middle points and are equal to ½OA. Therefore O′ is the centre of the circle DD′EE′FF′, and by definition the Brocard circle is the circle described on OK as diameter; therefore "le premier cercle de Lemoine et le cercle de Brocard sont concentriques."

9230. (Professor Haughton, F.R.S.)—Prove the following equation in Thermodynamics, and apply it to the subjoined example:—

$$JdQ = dU + pdv,$$

J = Joule's coefficient; Q = Quantity of heat; p = External pressure; v = Volume; U = Internal work.

Example.—A lead bullet strikes an iron target with a velocity of 1000 feet per second: find how much the temperature of the bullet rises, on the supposition that the target is perfectly rigid, the specific heat of lead being 0·031; and explain the extraordinary result at which you arrive.

Solution by H. J. WOODALL, A.R.C.S.

The whole quantity of work which disappears as one form of energy will reappear as another form, probably heat.

JdQ is the work put in, and equals that of the bullet moving at 1000 feet per second $= \frac{1}{2}mv^2 = 500,000\ m$ ft. lbs.; part, dU, goes to raising the temperature of the bullet; the rest, pdv, goes to increasing the volume of the bullet (included in the dU will be the heat required to vapourize the lead). $\therefore\ JdQ = dU + pdv$.

In the example, we will assume that the whole heat goes to raise the temperature; then we have

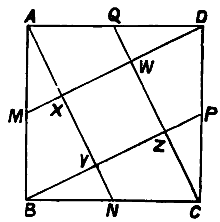
additional heat (in ft. lbs.) $= J.Q = 772 \times T \times 0{\cdot}031 \times m = m \times 500,000$;

$\therefore\ T = 500,000/(0{\cdot}031 \times 772) = 21,000°$ Fahr., which is absurd.

It is, however, probable that pdv is the larger of the two terms on the right-hand side, and not equal to zero as taken. But whatever the heat may be it will be shared by the target, and will be conducted away by the rapidly moving stream of air caused by the motion of the bullet.

11766. (EDITOR.)—If M, N, P, Q are the mid-points of the sides AB, BC, CD, DA of a square ABCD, prove that the intersections of the lines AN, BP, CQ, DM form a square which is one-fifth of the square ABCD.

Solution by W. J. CONSTABLE; J. DALE; *and others.*



To show XYZW is a square. MD, BP are parallel, for MB = and parallel to DP. So AN, QC are parallel. Then XYZW is a parallelogram.

Next, from equal triangles CQD, BPC,

angle ZPC = DQC = complement of ZCP;

therefore PZC is a right angle. Thus WZY is a right angle.

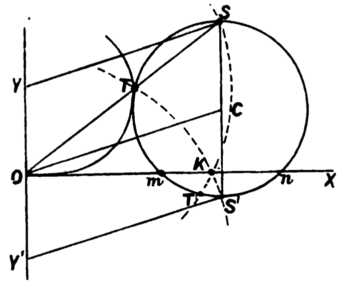
It may be shown that

$$BY = YZ = ZC.$$

Now square on BC = sum of squares on BZ, ZC
= five times square on YZ.

11730. (Professor B(?)NÉZECH.)—Sur une droite OX on prend deux points variables M, N, tels que $OM \cdot ON = k^2$. Par M et N on fait passer une circonférence C de rayon donné R; on trace ensuite une seconde circonférence tangente en O à OX et en T à la circonférence C. Lieu du point T.

Solution by J. C. St. Clair; R. Knowles, B.A.; *and others.*



Let C be the centre of any circle through m, n, and let $(\mathrm{OK})^2 = k^2$.

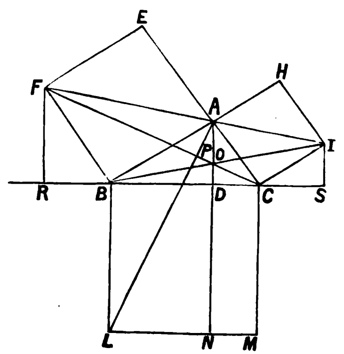
Inverting from O with radius OK, the circle C is unchanged, and the circle touching it at T becomes the tangent at S parallel to OX. On the perpendicular at O take OY = OY′ = CS. Then, since $(\mathrm{OC})^2 = (\mathrm{OK})^2 + (\mathrm{CS})^2$, the sides of the parallelogram YOCS are invariable, and the locus of S is a circle, centre Y and radius OC = YK. The inverse of this circle, *i.e.* the locus of T, is the equal circle having its centre at Y′; for the intercepts made by the two circles on OY, OY′ are equal and of opposite sign, and their product is $(\mathrm{OK})^2$. The circle Y itself is the locus of the points of contact whose inverses are S′ diametrically opposed to S.

11661. (Professor Van Aubel.)—Sean AEFB, AHIC los cuadrados construídos sobre los lados del ángulo recto de un triángulo ABC, rectángulo en A; O el punto de interseccíon de las rectas CF, BI; AOD la perpendicular bajada desde el vértice A sobre BC. Demonstrar que

(1) $1/\mathrm{AO} = 1/\mathrm{AD} + 1/\mathrm{BC}$; (2) $\mathrm{AB}.\mathrm{FC}.\mathrm{IO} = \mathrm{AC}.\mathrm{FO}.\mathrm{IB}$;

(3) $\mathrm{IO}/\mathrm{OB} = (\mathrm{AB}+\mathrm{AC})\,\mathrm{AC}/\mathrm{AB}^2$; (4) $\mathrm{FO}/\mathrm{OC} = (\mathrm{AB}+\mathrm{AC})\,\mathrm{AB}/\mathrm{AC}^2$.

Solution by J. Dale, M.A.; R. F. Davis, M.A.; *and others.*



(1) Construct the figure as directed; produce AD to meet the opposite side of the square on BC in N. Join AL; draw FAI. Then, from similar triangles, AP can be proved perpendicular to FC; hence PLNO is inscriptible in a circle, and

$$\mathrm{AO}.\mathrm{AN} = \mathrm{AP}.\mathrm{AL} = \mathrm{AP}.\mathrm{CF} = 2\Delta\mathrm{ACF} = 2\Delta\mathrm{ABC} = \mathrm{AD}.\mathrm{BC};$$

$$\therefore \frac{1}{\mathrm{AO}} = \frac{\mathrm{AN}}{\mathrm{BC}.\mathrm{AD}} = \frac{\mathrm{DN}+\mathrm{AD}}{\mathrm{BC}.\mathrm{AD}} = \frac{\mathrm{BC}+\mathrm{AD}}{\mathrm{BC}.\mathrm{AD}} = \frac{1}{\mathrm{AD}} + \frac{1}{\mathrm{BC}}.$$

(2) Draw FR, IS perpendicular to BC produced both ways. Then

$$\Delta FBR = \Delta ABD, \quad \Delta ICS = \Delta ADC, \quad \text{and} \quad BR = AD = CS,$$

$$\frac{IB}{IO} = \frac{BS}{DS} = \frac{BC+AD}{DS}, \qquad \frac{FO}{FC} = \frac{DR}{CR} = \frac{DR}{BC+AD};$$

$$\therefore \frac{IB}{IO}\cdot\frac{FO}{FC} = \frac{DR}{DS} = \frac{AF}{AI} = \frac{AB}{AC}; \quad \therefore AB.FC.IO = AC.FO.IB.$$

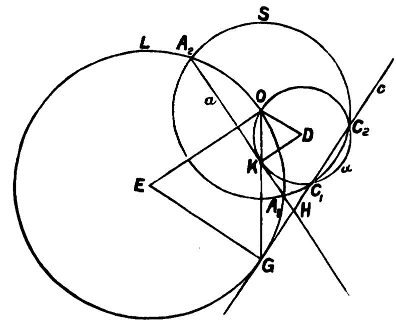
(3) $$\frac{IO}{OB} = \frac{SD}{BD} = \frac{AD+CD}{BD} = \frac{(AD+CD)BC}{AB^2}$$

$$= \frac{AD.BC+BC.CD}{AB^2} = \frac{AB.AC+AC^2}{AB^2} = \frac{(AB+AC)AC}{AB^2}.$$

Similarly, we have $$\frac{FO}{OC} = \frac{(AB+AC)AB}{AC^2}.$$

11707. (A. E. Jolliffe.)—From two points, B, C, tangents are drawn to a conic S, and these four tangents, together with the polars of B and C with respect to S, all touch a conic α. Similarly the pairs of points CA, AB determine the conics β and γ respectively. Prove, by pure geometry, that, if A lie on α, then B lies on β, and C on γ.

Solution by H. W. Curjel, B.A.



Reciprocate the figure so that the tangents to S from B become the circules; then B becomes b, the line at infinity, and α and γ become circles. Let A, C become the straight lines a, c.

In the reciprocated figure, let O be the centre of the circle S (pole of b). Let S cut c in C_1C_2 and a in A_1A_2. Then α is the circle OC_1C_2, and evidently passes through the pole of C_1C_2; let D be its centre, and let it touch a in K. Let a cut c in H. Join OK, cutting c in G. Draw OE, GE perpendicular to a and c. Then evidently $OE = EG$. With centre E draw the circle OGL. Then $\angle HGK = \frac{1}{2}\angle OEG = \frac{1}{2}\angle ODK = \angle HKG$; therefore $HG = HK$; therefore $HG^2 = HK^2 = HC_1 . HC_2$; therefore H is on the radical axis of S and OGL. Also HA_1A_2 or a is perpendicular to EO; therefore a is radical axis of S and OGL; therefore circle GLO passes through A_1A_2, and is therefore the circle γ, and it touches c at G. Hence if a touches α, c touches γ. Hence in the original figure, if A lies on α, C lies on γ, and similarly B lies on β.

11700. (Professor BÉNÉZÉCH.)—Soient O, O_a, O_b, O_c les centres des cercles inscrits et exinscrits d'un triangle ABC, rectangle en A. Si D est le milieu de l'hypoténuse, démontrer que $DO^2 + DO_a^2 = DO_b^2 + DO_c^2$.

Solution by Rev. D. THOMAS, M.A.; C. MORGAN, M.A.; *and others.*

It is known that

$$DO^2 = R^2 - 2Rr, \quad DO_a^2 = R^2 + 2Rr_a,$$

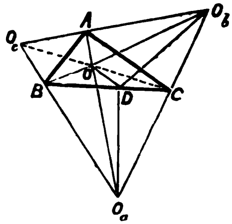
$$DO_b^2 = R^2 + 2Rr_b, \quad DO_c^2 = R^2 + 2Rr_c,$$

and $\quad r_a = s, \quad r = s - a,$

$$r_b = s(s-a)/(s-b), \quad r_c = s(s-a)/(s-c);$$

therefore $\quad r_b + r_c = a = r_a - r,$

and the above equality follows.



11639. (MORGAN BRIERLEY.)—Let CD be the diameter of a circle of centre O, AB a chord at right angles to CD, the point of intersection being M: on OM draw another circle, and from any point in its circumference draw a tangent TE to a point in the circumference of the outer circle, from which inflect lines to A and B; then prove that

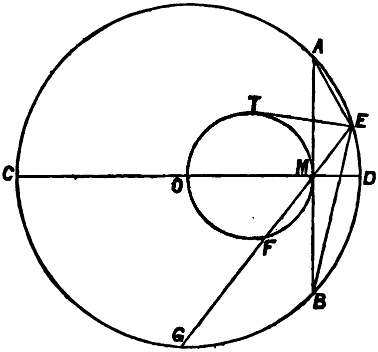
$$AE^2 + BE^2 = 4ET^2.$$

Solution by T. SAVAGE; H. W. CURJEL, B.A.; *and others.*

Produce EM to meet the circles OTM, ABC in F and G. Then OF is perpendicular to EG; therefore

GF = FE. Also,

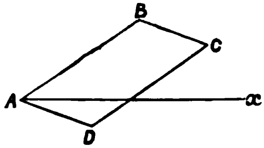
$$AE^2 + BE^2$$
$$= 2BM^2 + 2EM^2$$
$$= 2EM \cdot MG + 2EM^2$$
$$= 2EG \cdot EM$$
$$= 4EF \cdot EM$$
$$= 4ET^2.$$



11522. (Professor MANNHEIM.) — Soit ABCD un parallélogramme articulé. Le sommet A est fixe, et les côtés AB, AD tournent autour de

A d'angles égaux en sens inverses. Démontrer que le point C décrit une ellipse.

. *Solution by* H. W. Curjel, B.A.; Prof. Radikanshuan; *and others*.



Taking A as origin, and the position of AB when $\angle$ BAD is zero as axis of x, let AB $= a$, AD $= b$, $(x, y) =$ coordinates of C, and $\angle$ BA$x = \theta$; then

$$x = a\cos\theta + b\cos\theta,$$
$$y = a\sin\theta - b\sin\theta;$$

hence $\left(\dfrac{x}{a+b}\right)^2 + \left(\dfrac{y}{a-b}\right)^2 = 1$, and the locus of C is an ellipse.

8632. (Professor Haughton, F.R.S.) — Rosetti's formula for radiation is $y = a\mathrm{T}^2(\mathrm{T}-\theta) - b(\mathrm{T}-\theta)$, where $y =$ Thermal effect on galvanometer, T = absolute temperature of the hotter body, $\theta =$ absolute temperature of the colder body, a, b, constants to be found. Determine a and b from the first two of the following experiments, and from them calculate for comparison with the third experiment:—

No.	$\mathrm{T}-\theta$	Galvanometer.	Surrounding temperature.
1	172·8° C.	116·7	
2	232·8 ,,	204·0	= 23·8° C.
3	272·8 ,,	283·5	

Solution by H. J. Woodall, A.R.C.S.

Since $y = a\mathrm{T}^2(\mathrm{T}-\theta) - b(\mathrm{T}-\theta)$, therefore $y/(\mathrm{T}-\theta) = a\mathrm{T}^2 - b$.

$$(1)\ \ 116{\cdot}7/172{\cdot}8 = {\cdot}67[illegible] = a(172{\cdot}8 + 23{\cdot}8 + 273{\cdot}4)^2 - b$$
$$= a \times 470^2 - b;$$
$$(2)\ \ 204{\cdot}0/232{\cdot}8 = {\cdot}876 = a \times 530^2 - b;$$
$$(3)\ \ 283{\cdot}5/272{\cdot}8 = 1{\cdot}041 = a \times 570^2 - b.$$

From (1) and (2), $a = {\cdot}201/60000 = {\cdot}00000335$, $b = {\cdot}066$.

In (3), $a \times (570)^2 - b = 1{\cdot}1084 - {\cdot}066 = 1{\cdot}042$ (a result which differs in the last place only of decimals).

11732. (Professor Lucas.)—Le produit des 1000 premiers nombres est terminé par 249 zéros.

Solution by J. C. St. Clair, D. Biddle, *and others.*

The number of zeros is evidently the highest power of 5 contained in the product, and this is the sum of the quotients obtained by dividing 1000 by 5, 5^2, 5^3, 5^4, &c. The result is

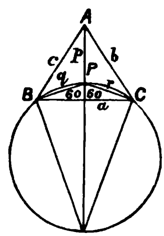
$$200+40+8+1 = 249.$$

[The product of the first thousand numbers is thus seen to have 250 final zeros less 1. In the case of the first million there are 250,000 less 2, and in the case of the first billion, 250,000,000,000 less 3.]

11605. (R. Chartres.)—If P be a point within the triangle ABC, whose centroid is G, and if PA, PB, PC be denoted by p, q, r, and a^2, b^2, c^2 by α, β, γ, find (1) P when $\dfrac{p-q}{\beta-\alpha} = \dfrac{q-r}{\gamma-\beta} = \dfrac{r-p}{\alpha-\gamma}$ = a maximum; (2) show that $(p+q+r)(\alpha+\beta+\gamma) = \dfrac{\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma}{p^3+q^3+r^3-3pqr}$; also (3) find the locus of P if A moves on a curve; (4) and the locus of A if

$$2\Sigma(\mathrm{PA}^2)-3\Sigma(\mathrm{GA}^2) = \text{a constant.}$$

Solution by the Proposer.



Let P be Fermat's point, then (1)

$$b^2-c^2 = (p^2+r^2+pr)-(p^2+q^2+pq)$$
$$= (r-q)(p+q+r),$$

$$\therefore \quad \frac{r-q}{\beta-\gamma} = \frac{1}{p+q+r} = \text{a maximum.}$$

(2) By proportion,

$$\frac{(r-q)^2+(q-p)^2+(p-r)^2}{(\beta-\gamma)^2+(\alpha-\beta)^2+(\gamma-\alpha)^2} = \frac{1}{(p+q+r)^2},$$

which is (2).

(3) As long as the base BC is fixed, P must lie on the arc BC.

(4) Again, by (1), $\quad a^2+b^2+c^2 = 2(p^2+q^2+r^2)+(pq+qr+rp)$,

but $\quad a^2+b^2+c^2 = 3\Sigma(\mathrm{GA}^2)$, $\quad$ if G be the centroid;

$\therefore \quad 3\Sigma(\mathrm{GA}^2)-2\Sigma(\mathrm{PA}^2) = pq+qr+rp$, $\quad$ which $\propto$ area ABC.

Therefore, if area be constant, the locus of A is a straight line parallel to BC.

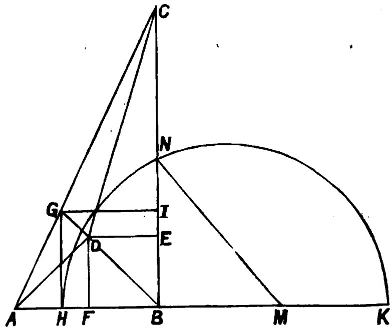
11741. (I. Arnold.)—Describe a square in a right-angled triangle having one angle of the square coincident with the right angle of the triangle, and such that the triangle formed by joining the extremities of the hypotenuse with the adjacent angle of the square shall be equal to the square.

Solution by the PROPOSER; C. BICKERDIKE; *and others.*

Let ABC be the given triangle, the angle B being a right-angle. Draw BG bisecting the angle B. Draw GH, GI perpendiculars on AB and BC. Produce AB and make

$$BK = \tfrac{1}{2}(AB + BC);$$

and on HK describe a semicircle cutting CB in N. Bisect BK in M, and join NM, and from M as centre, with MN as radius, describe an arc cutting BA in F; BF is the side of the required square.



$$BF^2 + FK^2 = 2BM^2 + 2MF^2\,(2MN^2) = 2BM^2 + BN^2.$$

But $FK^2 = (AB + BC)\,BF$, and $BF \cdot AB$ = double the triangle ADB. Similarly, $BF \cdot BC$ = double the triangle DBC; therefore

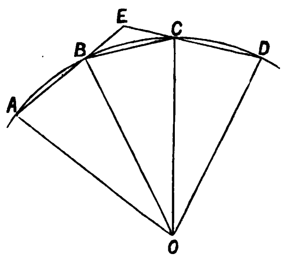
$$2BF^2 + BK^2 + \text{twice the figure } ABCD = 2BN^2 + BK^2;$$

$$\therefore\ 2BF^2 + 2ABCD = 2BN^2; \quad \therefore\ BF^2 + ABCD = BN^2 = HB \cdot BK.$$

But, HB being = HG = GI, therefore $HB \cdot BK$ = triangle ABC; therefore BF^2 = the triangle ADC.

11765. (Professor NILKANTHA SARKAR, M.A.)—Soient AB, BC, CD trois côtés consécutifs d'un polygone régulier, de centre O, et E le point d'intersection de AB et CD. Démontrer que les quatre points A, E, C, O sont sur une circonférence.

Solution by W. J. GREENSTREET, M.A.; H. W. CURJEL, B.A.; *and others.*



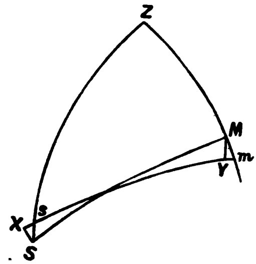
The angle BAO evidently = angle OCD; therefore A, O, C, E are concyclic.

11318. (C. MORGAN, R.N.)—In finding the longitude by lunar observation, if a, a' are the apparent altitudes of the observed body and

the moon, d the apparent distance, and x, y the corrections in altitude: show that an approximate correction (additive or subtractive) to be applied to the apparent distance to obtain the true is

$$\sin a' \left\{x \sec a \operatorname{cosec} d \mp y \sec a' \cot d\right\} \pm \sin a \left\{y \sec a' \operatorname{cosec} d \mp x \sec a \cot d\right\}.$$

Solution by the PROPOSER.



Let SM be the true lunar distance, M being the moon and S the body. Let sm be the apparent distance.

Draw SX, MY, arcs of great circles perpendicular to sm; then correction to apparent distance is approximately

$$sX \pm mY = x \cos SsX \pm y \cos Mmy$$

$$= x\frac{\cos Zm - \cos Zs \cos d}{\sin Zs \sin d}$$

$$\pm y\frac{\cos Zs - \cos Zm \cos d}{\sin Zm \sin d}$$

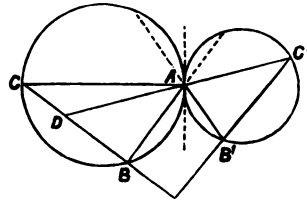
$$= x(\sin a' \sec a \operatorname{cosec} d - \tan a \cot d)$$

$$\pm y(\sin a \sec a' \operatorname{cosec} d - \tan a' \cot d),$$

which gives the stated result, the correction being additive or subtractive according to the value of the angles at s and m, and is easily seen from a figure.

11554. (Professor MANNHEIM.) — Deux circonférences Δ, Δ' se touchent au point A; deux droites rectangulaires rencontrent ces circonférences respectivement aux points B, C; B′, C′. Démontrer que le somme des angles aigus formés par les droites BAB′, CAC′ est égale à un angle droit.

Solution by G. G. MORRICE, M.D.; J. C. ST. CLAIR; *and others*.



Produce C′A to meet BC in D;

then $\angle ADB = C + CAD = \frac{1}{2}\pi - C'$;

$\therefore \quad C + C' + \text{sup. } CAC' = \frac{1}{2}\pi$;

i.e., $\angle BAB' + \text{sup. } CAC'$

$= $ a right angle.

Again, if we produce BA, B′A, it is evident that

sup. BAC′ + BAB′ + sup. B′AC

+ CAD = 2 right angles;

but $\angle BAB' + \angle CAD = 1$ right angle;

therefore sup. BAC′ + sup. B′AC = 1 right angle.

Hence B′ and C′ may be interchanged.

11539. (W. J. GREENSTREET, M.A.)—Two concentric ellipses have parallel axes. Q is the intersection of the polars of any point P with regard to the ellipses. Find the locus of Q if the locus of P is a straight line.

Solution by R. KNOWLES, B.A.; H. W. CURJEL, M.A.; *and others.*

Let hk be the coordinates of P, $x/m + y/n = 1$ the locus of P; then

$$h/m + k/n = 1 \quad \text{.................................(1)},$$

and $$a^2ky + b^2hx = a^2b^2, \quad a'^2ky + b'^2hx = a'^2b'^2 \text{...............(2, 3)},$$

the equations to the polars of P; eliminating hk between (1), (2), (3), we find the locus required to be

$$mn\,(a'^2b^2 - a^2b'^2)\,xy + b^2b'^2\,(a^2 - a'^2)\,mx - a^2a'^2\,(b^2 - b'^2)\,ny = 0,$$

which is a rectangular hyperbola through the origin with asymptotes parallel to the axes of x and y, and passing through the poles of the locus of P with respect to the ellipses.

10835. (Professor DE LONGCHAMPS.)—Un arc quelconque, pris sur une hyperbole équilatère, est vu, de deux points diamétralement opposés sur la courbe, sous le même angle. En déduire le théoreme suivant, facile à vérifier directement:—Soient A, A′ deux points diamétralement opposés sur une hyperbole équilatère H. Les circonférences passant par M tangentiellement à H, et respectivement par les points A, A′, sont égales.

Solution by W. J. GREENSTREET, M.A.

If the hyperbola be $xy = a^2$, and A, M, N be (x_1, y_1), (x_2, y_2), (x_3, y_3), $x_1y_1 = m^2$, $x_2y_2 = m^2$; therefore

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{-m^2}{x_1x_2} \text{ and } \frac{y_1 - y_3}{x_1 - x_3} = \frac{-m^3}{x_1x_3};$$

therefore $$\tan \text{MAN} = m^2x_1\,\frac{x_2 - x_3}{m^4 + x^2x_2x_3}, \ r \text{ similarly} = \tan \text{MA}'\text{N}.$$

The deduction at once follows when M and N coincide.

7177. (J. HAMMOND, M.A.)—Prove that, if $\phi(x) = \phi\left(\dfrac{cx}{1-x}\right)$.

$$\int_0^\infty \frac{\phi(x)}{x^2}\,dx = \frac{1}{c}\int_0^1 \phi(x)\,\frac{dx}{x^2}, \qquad \int_0^\infty \frac{\phi(x)\,dx}{x\,(x+c-1)} = \int_0^1 \frac{\phi(x)\,dx}{x\,(x+c-1)}.$$

Solution by H. W. Curjel, B.A.

Let $y = \frac{cx}{1-x}$; then $\phi(y) = \phi\left(\frac{cx}{1-x}\right) = \phi(x)$; $x = \frac{y}{y+c}$;

$$\therefore \quad \frac{dx}{x^2} = \frac{c\,dy}{y^2} \text{ and } \frac{dx}{x(x+c-1)} = \frac{c\,dy}{(y+c)^2 \frac{y}{y+c}\left(\frac{y}{y+c}+c-1\right)}$$
$$= \frac{dy}{y(y+c-1)}.$$

Also, when $x = 0$, $y = 0$; and, when $x = 1$, $y = \infty$;

$$\therefore \quad \frac{1}{c}\int_0^1 \phi(x)\frac{dx}{x^2} = \int_0^\infty \frac{\phi(y)}{y^2}dy, \text{ and } \int_0^1 \frac{\phi(x)\,dx}{x(x+c-1)} = \int_0^\infty \frac{\phi(y)\,dy}{y(y+c-1)}.$$

11742. (J. O'Byrne Croke, M.A.)—Arrange the simple factors of the expression $n(n^2-1^2)(n^2-2^2)(n^2-3^2)(n^2-4^2)(n^2-5^2)\dots(n^2-12^2)$ in a magic square of twenty-five compartments.

Solution by Professor Zerr; H. J. Woodall, A.R.C.S.; *and others.*

The magic square of 25 compartments with the given factors as elements may be written as annexed in the margin.

$n+10$	$n-6$	$n+7$	$n-12$	$n+1$
$n-4$	$n+8$	$n-11$	$n+5$	$n+2$
$n-3$	$n-9$	n	$n+9$	$n+3$
$n-2$	$n-5$	$n+11$	$n-8$	$n+4$
$n-1$	$n+12$	$n-7$	$n+6$	$n-10$

Here the sum of the quantities in any row or column, or in either of the diagonal ranges will be found to be equal to $5n$.

[From Mr. Woolhouse's solution to his Quest. 11460 (Vol. LVII., p. 104) the particular result follows by writing $n-12 = 1$, &c.]

11665. (Professor Morel.)—Une droite AB, de longueur donnée l, se meut entre deux droites fixes CX, CY. Démontrer que le centre du cercle circonscrit au triangle CAB et l'orthocentre décrivent des circonférences.

Solution by R. F. Davis, M.A.; R. Knowles, B.A.; *and others.*

Let the angle XCY be denoted by ω. Then the distance of the centre O of the circle circumscribed to the triangle CAB = radius of this circle = $\frac{1}{2}l \operatorname{cosec} \omega$; while the distance of the orthocentre = twice perpendicular from O on AB = $l \cot \omega$. Both of these values are constant, &c.

2443. (J. GRIFFITHS, M.A.)—Prove that the Jacobian of the three conics represented by the trilinear equations

$$S = \sin^2 A \,.\, \alpha^2 + \&c. - 2\sin B \sin C \,.\, \beta\gamma - \&c. = 0,$$
$$S' = \cos^2 A \,.\, \alpha^2 + \&c. - 2\cos B \cos C \,.\, \beta\gamma - \&c. = 0,$$
$$F = \sin 2A \,.\, \alpha^2 + \&c. - 2\sin A \,.\, \beta\gamma - \&c. = 0,$$

breaks up into the three right lines

$$\frac{\beta}{\sin(C-A)} + \frac{\gamma}{\sin(A-B)} = 0, \qquad \frac{\gamma}{\sin(A-B)} + \frac{\alpha}{\sin(B-C)} = 0,$$
$$\frac{\alpha}{\sin(B-C)} + \frac{\beta}{\sin(C-A)} = 0.$$

Hence show how to construct geometrically the common self-conjugate triangle of the three conics in question.

Solution by H. J. WOODALL, A.R.C.S.

The Jacobian is

$$\begin{vmatrix} \alpha\sin^2 A - \beta\sin A\sin B - \gamma\sin A\sin C, & -\cos A(-\alpha\cos A + \beta\cos B + \gamma\cos C), & \alpha\sin 2A - \beta\sin C - \gamma\sin B \\ -\sin B(\alpha\sin A - \beta\sin B + \gamma\sin C), & -\cos B(\alpha\cos A - \beta\cos B + \gamma\cos C), & -\alpha\sin C + \beta\sin 2B - \gamma\sin A \\ -\sin C(\alpha\sin A + \beta\sin B - \gamma\sin C), & -\cos C(\alpha\cos A + \beta\cos B - \gamma\cos C), & -\alpha\sin B - \beta\sin A + \gamma\sin 2C \end{vmatrix} = 0;$$

which reduces to

$$\left(\frac{\beta}{\sin(C-A)} + \frac{\gamma}{\sin(A-B)}\right)\left(\frac{\gamma}{\sin(A-B)} + \frac{\alpha}{\sin(B-C)}\right)\left(\frac{\alpha}{\sin(B-C)} + \frac{\beta}{\sin(C-A)}\right) = 0;$$

i.e., the locus consists of these three straight lines.

It thence appears that these three lines are the sides of the required triangle. [See SALMON's *Conics*, § 388.]

7182. (R. KNOWLES, B.A., L.C.P.)—Solve the symbolical equations

$$Du + (D-n-1)^2 \epsilon^\theta u = 0.$$

Solution by Professor SEBASTIAN SIRCOM.

Putting $\epsilon^\theta = x$ and $x\dfrac{d}{dx} = D$, we have $(D-n-1)^2 xu + Du = 0$.

Assuming $u = \Sigma a_m x^m$, we have $(m-n-1)^2 u_{m-1} + ma_m = 0$; this gives the solution in finite terms,

$$u = A\left(1 - n^2 x + n^2(n-1/n)^2 x^2 - \dots + (-1)^n n!\, x^n\right) = Av;$$

then, taking $u = vw$, we have for w,

$$\left(x^2\frac{d^2w}{dx^2} - (2n-1)\,x\frac{dw}{dx} + \frac{dw}{dx}\right) + 2x^2\frac{dv}{dx}\cdot\frac{dw}{dx} = 0, \text{ whence}$$

$$w = c\int\frac{x^{2n-1}\,e^{1/x}}{v^2}\,dx + \mathrm{G}, \text{ and } y = \mathrm{A}v + \mathrm{B}v\int\frac{x^{2n-1}\,e^{1/x}}{v^2}\,dx.$$

Also, assuming $u = \mathrm{A}v + \mathrm{B}\,(v\log x + w_1)$, we have

$$(\mathrm{D}-n-1)^2\,xw_1 + \ldots + \mathrm{D}w_1 + 2\,(\mathrm{D}-n-1)\,xv + v = 0,$$

and w_1 can be obtained in the form of a series consisting of positive powers of x up to the nth, and the infinite series

$$-1/(n+1)^2\,x - 1/(n+1)^2\,(n+2)^2\,x^2 \ldots .$$

Also thus:—Operate on the given equation with

$$(\mathrm{D}-1)\,(\mathrm{D}-2)\ldots(\mathrm{D}-n);$$

then, since

$$(\mathrm{D}-1)\,(\mathrm{D}-2)\ldots(\mathrm{D}-n-1)\,xu = x\,.\,\mathrm{D}\,(\mathrm{D}-1)\ldots(\mathrm{D}-n)\,u,$$

we have, by putting

$$\mathrm{D}\,(\mathrm{D}-1)\ldots(\mathrm{D}-n)\,u \equiv x^{n+1}\left(\frac{d}{dx}\right)^{n+1}u = v, \quad (\mathrm{D}-n-1)\,xv + v = 0;$$

whence $$v = cx^n\,e^{1/x}; \quad u = c\iiint\ldots\frac{e^{1/x}}{x}\,(dx)^{n+1},$$

retaining only one more constant.

11624. (Professor CHAKRIVARTI.)—If on a straight line of length $a+b$ be measured at random two lengths a, b, the probability that the common part of these lengths shall not exceed c is c^2/ab, ($c < a$ or b); and the probability of the smaller b lying entirely within the larger a is $(a-b)/a$.

Solution by H. W. CURJEL, B.A.; Professor ZERR; *and others.*

Chance that the common part is less than c = (chance that common is less than c and that part of the length b is to the left of the length a) × 2

$$= 2\int_{b-c}^{b}\{x-(b-c)\}\,dx\bigg/\int_0^a b\,dx = \frac{c^2}{ab}.$$

The chance that b lies within $a = a - b/a$, for the left end of b has a range $= a$, the part $a-b$ to the right of the left end of a being favourable.

11739. (D. BIDDLE.)—A random particle strikes an irregular tetrahedron. Find the probability that it strikes a particular side.

Solution by Profs. ZERR, BHATTACHARYA, *and others*.

Let A, B, C, D be the areas of the sides; then

$$\frac{A}{A+B+C+D},\quad \frac{B}{A+B+C+D},\quad \frac{C}{A+B+C+D},\quad \frac{D}{A+B+C+D}$$

are the chances that it strikes sides A, B, C, D.

[The PROPOSER remarks that this is true of the randomest particle, but that where a random particle strikes under the influence of attraction, the probability is proportional to the solid angle subtended by the particular side from the centroid of the tetrahedron.]

11769. (E. WHITE.)—If a be a root of one of the equations

$$f(x) = 0,\quad \frac{df}{dx} = 0,\quad \frac{d^2f}{dx^2} = 0,$$

prove that (1) $f(3a) = 1$, where $f(x) \equiv 1 + \frac{x^3}{3\,!} + \frac{x^6}{6\,!} + \frac{x^9}{9\,!} + \dots$;

and (2) in general, $f(3x) = 1 + 9\,.\,f(x)\,.\,\frac{df}{dx}\,.\,\frac{d^2f}{dx^2}$.

Solution by H. W. CURJEL, B.A.; E. WHITE; *and others*.

Let $\omega^2+\omega+1 = 0$; then $f(x) = \frac{1}{3}(e^x+e^{\omega x}+e^{\omega^2x})$;

$$\therefore\quad 9f(x)\frac{df}{dx}\frac{d^2f}{dx^2}+1$$

$$= \tfrac{1}{3}\left\{(e^x+e^{\omega x}+e^{\omega^2x})(e^x+\omega e^{\omega x}+\omega^2e^{\omega^2x})(e^x+\omega^2e^{\omega x}+\omega e^{\omega^2x})+3\right\}$$

$$= \tfrac{1}{3}(e^{3x}+e^{3\omega x}+e^{3\omega^2x}-3e^{(1+\omega+\omega^2)x}+3)$$

$$= \tfrac{1}{3}(e^{3x}+e^{3\omega x}+e^{3\omega^2x}) = f(3x);\qquad \therefore\quad f(3a) = 1.$$

8846. (By Prof. ORCHARD, B.Sc., M.A.)—Find the *negative root* of the quadratic of which the positive root is $\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}$.

Solution by H. J. WOODALL, A.R.C.S.

Putting $x = \frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+x}$, we get $7x^2+8x-3 = 0$.

The positive root is between $\frac{1}{3}$ and $\frac{1}{4}$.

The negative root is between -2 and -1. Then, by Lagrange's

method of approximation, put $x = -2+1/y$ and transform, whence we get

$$\begin{array}{llll} & 9y^2-20y+7 = 0, \text{ which has root between} & 1 \text{ and } 2; & \text{put } y = 1+1/z, \\ \therefore & 4z^2+2z-9 = 0, \quad ,, \quad ,, & 1 \text{ and } 2; & ,, \ z = 1+1/w, \\ \therefore & 3w^2-10w-4 = 0,^* \quad ,, \quad ,, & 3 \text{ and } 4; & ,, \ w = 3+1/u, \\ \therefore & 7u^2-8u-3 = 0, \quad ,, \quad ,, & 1 \text{ and } 2; & ,, \ u = 1+1/v, \\ \therefore & 4v^2-6v-7 = 0, \quad ,, \quad ,, & 2 \text{ and } 3; & ,, \ v = 2+1/s, \\ \therefore & 3s^2-10s-4 = 0^* \text{ recurs}; & & \end{array}$$

$$\therefore \quad x = -2+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\underset{*}{3+}}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{\underset{*}{3+}}.$$

[For Method, see Todhunter's *Theory of Equations*, p. 137, &c.]

11325. (Rev. G. H. HOPKINS, M.A.)—From any point on the surface of a circular cylinder planes are drawn. Find the equation to the surface upon which the foci of all the elliptic sections are placed; and prove that the section of this surface by a plane through the fixed point, and containing the axis of the cylinder, will be the Logocyclic Curve.

Solution by H. W. CURJEL, B.A.

Take the fixed point as origin, and let the equations to the cylinder and plane through the origin be

$$x^2+z^2-2bx = 0; \quad lx+my+nz = 0, \text{ where } l^2+m^2+n^2 = 1.$$

The foci of the section are the feet of perpendiculars from points on the axis of the cylinder whose distances from the plane $= \pm b$. (Since the spheres inscribed in the cylinder and touching the plane touch it in the foci of the section.)

Hence the foci are given by $lx+my+nz = 0$, and

$$\frac{x-b}{l} = \frac{y+\{(lb\pm b)/m\}}{m} = \frac{z}{n} = \frac{my+lb\pm b}{m^2}$$

$$= \frac{lx-lb+my+lb\pm b}{l^2+m^2} = \frac{-nz\pm b}{1-n^2},$$

$$\therefore \ n = \pm\frac{z}{b}, \quad \therefore \ l = \pm\frac{x-b}{b}, \quad \therefore \ m = \frac{-nz-lx}{y} = \mp\frac{x(x-b)+z^2}{by}.$$

$$\therefore \quad \text{locus of foci is} \qquad z^2+(x-b)^2+\frac{\{x(x-b)+z^2\}^2}{y^2} = b^2,$$

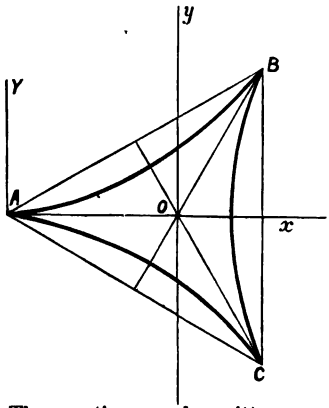
$$\text{i.e.} \qquad y^2(z^2+x^2-2bx)+\{x(x-b)+z^2\}^2 = 0.$$

Putting $z = 0$, and dividing by x, we get, as equation to the required section,
$$x(x-b)^2+y^2x-2by = 0;$$
or $r = b\cot\frac{1}{2}\theta$, taking the axis of y as initial line; this is the Logocyclic Curve.

11733. (Professor PICQUET.)—Construire la courbe

$$(x^2+y^2)^2+8\lambda x^3-24\lambda xy^2+18\lambda^2(x^2+y^2)-27\lambda^4=0.$$

Solution by H. W. CURJEL, B.A.



Transforming to polar coordinates, and writing the equation

$$r^4+32\lambda r^3\cos\theta\cos(\theta+\tfrac{1}{3}\pi)$$
$$\times\cos(\theta-\tfrac{1}{3}\pi)+18\lambda^2r^2-27\lambda^4=0,$$

we see that the curve is unaltered by turning it about the origin O through an angle $\frac{2}{3}\pi$; also the axis of x is evidently an axis of symmetry.

Again, transforming to the origin A $(-3\lambda, 0)$, the equation becomes

$$(X^2+Y^2)^2-4X^3\lambda-36XY^2\lambda$$
$$+108Y^2\lambda^2=0,$$

the approximation at the origin being $X^3=27y^2\lambda$. There is, therefore, a cusp at A, with the axis of X as cuspidal tangent; also we see that X must be positive. The equation may be written

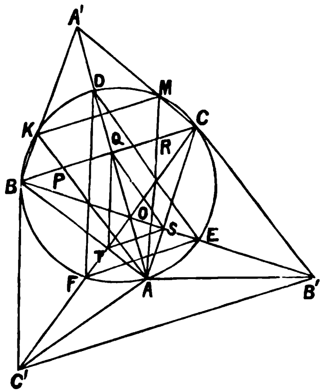
$$Y^2=18X\lambda-X^2-54\lambda^2\pm2(9\lambda-2X)^{\frac{3}{2}};$$

therefore X cannot be greater than $4\frac{1}{2}\lambda$. $Y=0$ gives $X=4\lambda$ and $dY/dX=\infty$; $X=3\lambda$ gives $Y^2=\left\{(108)^{\frac{1}{2}}-9\right\}\lambda$, therefore $Y=\lambda 1\cdot18$, about; $X=4\lambda$ gives $Y^2=0$ or $4\lambda^2$; at $X=4\lambda$, $y=2\lambda$, $dY/dX=1$.

Hence we see that the curve is symmetrically inscribed in the equilateral triangle ABC with A as vertex and $x=\frac{3}{2}\lambda$ as base, and has cusps at A, B, C; the perpendiculars on the opposite sides are the cuspidal tangents, and cut the curve again at right angles at distances $=4\lambda$ from the cusps.

9733. (R. TUCKER, M.A.)—ABC is a triangle; A′, B′, C′ are the images of A, B, C with respect to BC, CA, AB. The circumcircle ABC cuts A′BC (say) in K (on A′B), M (on A′C), and AK, AM, AA′ cut BC in P, R, Q respectively. Prove that (1) the orthocentres of the associated triangles lie on circle ABC; (2) triangle AKM has its sides parallel to and equal twice the sides of the pedal triangle of ABC, and is also equal triangle formed by the above-named orthocentres; (3) $CP.a=b^2$, $BR.a=c^2$, $AP.a=AR.a=bc$, $BP.a=a^2-b^2$, $CR.a=a^2-c^2$, *i.e.*, $PR.a=2bc\cos A$; (4) hence BA touches circle ARC, which contains a Brocard point of ABC; similarly for CA and circle APB; (5) $BR.CR'.AR''=abc=CP.BP'.AP''$ (where R′, R″, P′, P″ correspond to RP, on CA, AB respectively); K, K′ are the Brocard constants ($K=a^2+b^2+c^2$) of ABC, A′B′C′; then $K'-K=16\Delta^2/R^2$.

Solution by Professor G. B. M. Zerr.



(1) Since the triangles A′BC, B′AC, C′AB are equal to ABC, and since A′, B′, C′ are images of A, B, C, the orthocentres of the triangles are images of the orthocentre of ABC with respect to its sides.

But $BS.SE = AS.SC$,

or $a \sin C . SE = a \cos C . c \cos A$;

$\therefore\ SE = c \cos A \cot C = SO$;

also $DQ = b \cos C \cot B = QO$,

$TF = a \cos B \cot A = TO$;

therefore D, E, F are the orthocentres of the triangles.

(2) Arc KDC = arc CEA, both measure of $\angle B$, arc DC = arc CE both measure of $\angle(\frac{1}{2}\pi - C)$; therefore arc KD = arc AE, arc MKB = arc BFA both measure of $\angle C$, arc DB = arc BF both measure of $\angle(\frac{1}{2}\pi - B)$; arc DM = arc FA; therefore

arc KBA = arc DBF, arc DCE = arc MEA, arc KDM = arc FAE;

$\therefore\ DF = KA = 2QT,\quad DE = MA = 2QS,\quad KM = FE = 2TS.$

Also KA is parallel to DE is parallel to QS, DF is parallel to MA is parallel to QT; $\Delta AKM = \Delta DEF$ (three sides of one equal three sides of other).

(3) From the triangle PAC, since $\angle PAC = \angle B$ and $\angle P = \angle A$,

we have $\qquad CP \sin A = b \sin B, \quad$ or $\quad CP . a = b^2$.

Similarly, from the triangle BAR, we have

$$BR \sin A = c \sin C, \quad \text{or} \quad BR . a = c^2,$$

since $\qquad AP = PR, \quad AP . a = AR . a.$

But $\qquad AP \sin A = b \sin C, \quad$ or $\quad AP . a = AR . a = bc,$

$$BP . a = (a - CP)\, a = a^2 - CP . a = a^2 - b^2,$$

$$CR . a = (a - BR)\, a = a^2 - BR . a = a^2 - c^2,$$

$$PR \sin A = AP \sin(\pi - 2A) = 2AP \sin A \cos A;$$

therefore $\qquad PR . a = 2AP . a \cos A = 2bc \cos A.$

(4) Since $BR . BC = AB^2$, AB touches the circle through ARC at A: therefore one of the Brocard points is on this circumference. Since $CP . CB = CA^2$, CA touches the circle through APB at A, which contains the other Brocard point.

(5) $BR = c^2/a,\ CR' = a^2/b,\ AR'' = b^2/c,\ CP = b^2/a,\ BP' = a^2/c,$

$$AP'' = c^2/b; \quad \therefore\ BR . CR' . AR'' = CP . BP' . AP'' = abc;$$

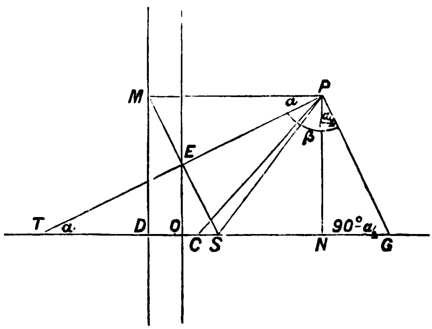
$$B'C'^2 = c^2 + b^2 - 2bc \cos 3A = a^2 + 8bc \cos A \sin^2 A,$$

$$A'C'^2 = b^2 + 8ac \cos B \sin^2 B, \quad A'B'^2 = c^2 + 8ab \cos C \sin^2 C;$$

$$\therefore\ K'-K = 8\left\{bc\cos A\sin^2 A + ac\cos B\sin^2 B + ab\cos C\sin^2 C\right\}$$
$$= 32\Delta^2\left\{\cos A/bc + \cos B/ac + \cos C/ab\right\}$$
$$= (16\Delta^2/a^2b^2c^2)\,\Sigma\,(2a^2b^2 - a^4) = 256\Delta^4/a^2b^2c^2 = 16\Delta^2/R^2.$$

7306. (Professor HUDSON, M.A.)—From a point P on a parabola focus S, PM and PN are drawn perpendicular to the directrix and axis; PT, PG are the tangent and the normal limited by the axis; what line represents the resultant of forces represented by PM, PT, PS, PN, PG?

Solution by Professor LAMPE.



Let $2p$ be the latus rectum. The two forces PM and PS give as resultant again PT. Thus we have to find the resultant of forces 2PT, PN, PG. Taking PT and PG as the two axes of reference, the three forces give as components, in direction of PG,

$$PG + PN\cos\alpha = X,$$

in direction of PT,

$$2PT + PN\sin\alpha = Y.$$

From $NG = p$, we get $PN = p\cot\alpha$, $PG = p/\sin\alpha$, $PT = p\cos\alpha/\sin^2\alpha$. Substituting in X and Y, we get

$$X = \frac{p}{\sin\alpha}(1+\cos^2\alpha),\quad Y = \frac{p\cos\alpha}{\sin^2\alpha}(2+\sin^2\alpha).$$

Let PC (see figure) be the direction of the resultant

$$R = (X^2+Y^2)^{\frac{1}{2}},\quad \text{and}\quad GPC = \beta,$$

then
$$\cot\beta = \frac{X}{Y} = \tan\alpha\,\frac{1+\cos^2\alpha}{2+\sin^2\alpha},\quad \sin\beta = \frac{Y}{R}.$$

The triangle PGC gives

$$GC = PG\frac{\sin\beta}{\cos(\alpha-\beta)} = \frac{p}{\sin\alpha(\cos\alpha\cot\beta + \sin\alpha)},$$

or, making use of the value calculated for $\cot\beta$, this becomes

$$GC = \frac{p}{4\sin^2\alpha}(2+\sin^2\alpha).$$

Now we have
$$\frac{1}{\sin^2\alpha} = 1 + \cot^2\alpha = 1 + \frac{y^2}{p^2} = 1 + \frac{2x}{p};$$

whence $\qquad CG = \dfrac{p}{2\sin^2\alpha} + \tfrac{1}{4}p = \tfrac{3}{4}p + x, \quad OC = \tfrac{1}{4}p = CS;$

moreover, $\qquad \dfrac{PC}{GC} = \dfrac{\cos\alpha}{\sin\beta}, \quad PC = \dfrac{R}{Y}\cos\alpha \,.\, GC;$

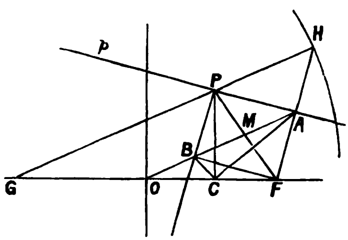
hence, substituting the trigonometric expressions for Y and GC, we at last obtain $PC = \frac{1}{4}R$.

Bisect OS in C and join CP; PC will be the direction of the resultant R; $4PC = R$.

[*Otherwise*:—Let A be the vertex; let the directrix cut SA in D; bisect AS in C. Then, by the parallelogram of forces, the resultant of forces represented by PM and PN is represented by PD, of PD and PS by twice PA, of PT and PG by twice PS, and of twice PA and twice PS by four times PC. Therefore, if PC be produced to K so that $PK = 4PC$, PK will represent the resultant of forces represented by PM, PT, PS, PN, PG. A Solution by Mr. Woodall is given on p. 59 of Vol. LVIII.]

11797. (Professor Tissot.)—Si l'on projette un foyer d'une conique sur la tangente et sur la normale en un point de la courbe, et ce dernier point sur l'axe focal, démontrer que (1) les deux premières projections seront en ligne droite avec le centre; (2) de la troisième, leur distance sera vue sous un angle droit; (3) dans l'angle formé avec l'axe par la droite qu'elles déterminent, la normale et la droite joignant la deuxième projection à la troisième seront anti-parallèles; (4) les distances du centre à ces deux dernières projections seront entre elles dans un rapport égal à l'excentricité, d'où résulte, sans calcul, le rapport connu de la différence ou de la somme des rayons vecteurs d'un point d'une conique avec la distance de ce point au second axe.

Solution by Professor Schoute; J. C. St. Clair; *and others.*



Représentons par F et G les foyers de la conique, par H un point du cercle directeur à centre G, et par p la droite normale à FH au point milieu A. Cette droite touche la conique au point P de GH. Soit B la projection de F sur la normale en P, et C celle de P sur l'axe FG.

(1) La droite OA est parallèle à GH, O et A étant les milieux de FG et FH. Donc AO et PF, AB et PF sont deux couples de droites antiparallèles par rapport à p. En d'autres termes, AB passe par O.

(2) Le cercle circonscrit au rectangle PAFB passe par C. Donc l'angle ACB est droit.

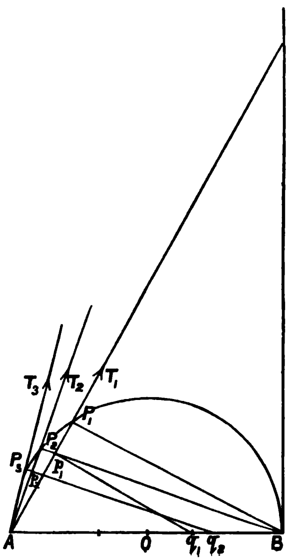
(3) Dans le cercle indiqué les arcs AP et BF sont égaux. On a donc
$$\angle PBA = \angle OCB.$$

(4) On a $OA.OB = OC.OF$, ou $OF/OA = OB/OC$. Donc $OB = e.OC$, si e représente l'eccentricité numérique. Parceque $OB = OM - MF = \frac{1}{2}(GP - FP)$, la dernière remarque se trouve vérifiée.

10980. (D. Biddle.)—On the straight line AB, with mid-point O, describe the semicircle APB. With centre A and radius AO, describe an arc cutting the semicircle in P_1. Join AP_1, BP_1, and between AP_1 and AB draw p_1q_1 parallel to BP_1, making $Ap_1 = Bq_1$. Again, with centre A and radius Ap_1, describe an arc cutting the semicircle in P_2; join AP_2, BP_2, and between AP_2 and AB draw p_2q_2 parallel to BP_2, making $Ap_2 = Bq_2$. Repeat the process indefinitely, and produce AP_1, AP_2, AP_3, &c. to meet the perpendicular to AB (at B) in T_1, T_2, T_3, &c. Prove that AT_1, AT_2, AT_3, &c. are successive multiples of AB, of which AP_1, AP_2, AP_3, &c. are the reciprocals, and find the mean of n of the series last-named, as represented by Bq_1, Bq_2, Bq_3, &c.

Solution by H. J. Wocdall, A.R.C.S.

Since p_1q_1 is parallel to P_1B, we have
$$Ap_1 : Aq_1 = AP_1 : AB = 1 : 2;$$
but $$Ap_1 = q_1B;$$
and therefore $= \frac{1}{3}AB$. So, again,
$$Ap_2 : Aq_2 = AP_2 : AB = 1 : 3;$$
but $$Ap_2 = q_2B;$$
and therefore $= \frac{1}{4}AB$. Continuing, we find
$$Ap_n = AB/(n+2) = AP_{n+1}.$$
Again, triangles AP_1B, ABT_1 are similar; therefore
$$AT_1 : AB = AB : AP_1 = 2 : 1;$$
hence, generally
$$AT_{n+1} : AB = AB : AP_{n+1} = n+2 : 1.$$
That is to say, AT_1, AT_2 are successive multiples of AB. Bq_1, Bq_2, Bq_3 ... are in harmonic progression. The *harmonic* mean of n of the series will be the middle term.



11090. (Professor MALILAL MALLIK, M.A.)—Find the value of

$$\frac{1}{2^2}\cdot\frac{1}{3^2}\cdot\frac{1}{4^2}\cdots\frac{1+2^4}{1+2^2}\cdot\frac{1+3^4}{1+3^2}\cdot\frac{1+4^4}{1+4^2}, \text{ \&c.}$$

Solution by Professor SEBASTIAN SIRCOM.

The given expression

$$= \{(1+1/2^4)(1+1/3^4)(1+1/4^4)\}\ldots / \{(1+1/2^2)(1+1/3^2)(1+1/4^2)\}.$$

We have $\sinh\pi = \pi(1+1)(1+1/2^2)(1+1/3^2)\ldots.$

Also $\sin\theta\sinh\theta = \theta^2(1-\theta^4/\pi^4)(1-\theta^4/2^4\pi^4)(1-\theta^4/3^4\pi^4)\ldots,$

then $\sin\frac{1+i}{\sqrt{2}}\theta\sinh\frac{1+i}{\sqrt{2}}\theta = i\theta^2(1+\theta^4/\pi^4)(1+\theta^4/2^4\pi^4)\ldots.$

Whence, putting $\theta = \pi$, and simplifying,

$$\tfrac{1}{2}\{\cosh(\sqrt{2}\,\pi)-\cos(\sqrt{2}\,\pi)\} = \pi^2(1+1)(1+1/2^4)(1+1/3^4)\ldots,$$

and the value required is $\{\cosh(\sqrt{2}\,\pi)-\cos(\sqrt{2}\,\pi)\}/2\pi\sinh\pi$.

11785. (Professor ZERR.)—A bucket and a counterpoise connected by a string passing over a pulley just balance one another; the bucket is at a distance h from the ground, and an elastic ball is dropped into the centre of the bucket from a distance h above it: find (1) the elasticity of the ball so that the bucket may reach the ground just as the ball ceases to rebound; and (2) the time it takes, the masses of ball and bucket being equal.

Solution by Rev. T. R. TERRY; W. J. DOBBS, M.A.; *and others.*

Let M be the mass of bucket or counterpoise, and m the mass of the ball. Velocity of ball just before first impact is $(2gh)^{\frac{1}{2}}$. The velocity of the ball relative to the bucket has to be gradually destroyed by impacts. Therefore total time between ball starting from rest and settling down to rest on bucket is

$$\left(\frac{2h}{g}\right)^{\frac{1}{2}}(1+2e+2e^2+\ldots) = \frac{1+e}{1-e}\left(\frac{2h}{g}\right)^{\frac{1}{2}}.$$

During this time a force mg acts; therefore total momentum at end

$$= m\,\frac{1+e}{1-e}(2gh)^{\frac{1}{2}}.$$

Loss of kinetic energy by an impact $= \dfrac{Mm}{2M+m}(1-e^2)$ (rel. vel. before impact)^2;

therefore total loss of K.E. by impacts $= (2Mmgh)/(2M+m)$.

By the equation of energy,

total energy imparted to the system = K.E. at end + K. E. lost; therefore

$$2mgh = \left(\frac{1+e}{1-e}\right)^2 \frac{m^2gh}{2M+m} + \frac{2Mmgh}{2M+m}; \quad \therefore\ e = \frac{(2M+2m)^{\frac{1}{2}} - m^{\frac{1}{2}}}{(2M+2m)^{\frac{1}{2}} + m^{\frac{1}{2}}}$$

In the particular case where $m = M$, $e = \frac{1}{3}$, and time $= 2\left(\frac{2h}{g}\right)^{\frac{1}{2}}$.

11530. (Rev. C. L. Dodgson, M.A.)—Required a general investigation of the following trigonometrical formula, which is useful in calculating limits for the value of π. The problem which I set myself was to break up $\tan^{-1} 1/a$ into two angles of the same form. Let

$$\tan^{-1}\frac{1}{a} = \tan^{-1}\frac{1}{a+x} + \tan^{-1}\frac{1}{a+y} = \tan^{-1}\frac{2a+x+y}{a^2+a(x+y)+xy-1}.$$

Then, if $(xy-1)$ were made equal to a^2, the denominator would become $a(2a+x+y)$; *i.e.*, the fraction would become $1/a$. Hence we get the rule: Let $(a^2+1) = xy$; *i.e.*, break up (a^2+1) into any two factors, call them x and y, and use them in the formula with which we began. Thus, if $a = 3$, $a^2+1 = 10 = 2 \times 5$. Hence $\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$. By the use of this formula, I have obtained 3·141597 and 3·141583 as limits for π.

Solution by H. J. Woodall, A.R.C.S.

If $\tan^{-1}1/a = \tan^{-1}1/(a+x) + \tan^{-1}1/(a+y)$, or, as I prefer to write it (in Gauss's notation), $\text{A}\cot a = \text{A}\cot(a+x) + \text{A}\cot(a+y)$, then

$$\text{A}\cot a = \text{A}\cot\left[\{a^2+a(x+y)+xy-1\}/\{2a+x+y\}\right];$$

and we get, with the assumption that the angles are the smallest possible,

$$2a^2+a(x+y) = a^2+a(x+y)+xy-1; \text{ whence } a^2+1 = xy \ \ldots\ (C_2).$$

This useful formula occurs in the 7th and 8th editions of the *Ency. Brit.*, Art. "Algebra" (Ed. 7, Vol. II., p. 497, Ed. 8, Vol. II., p. 557, both by Prof. Wallace). The series of relations which I obtained by the aid of $a^2+1 = xy$ reminds me of a remark by Prof. Chrystal in his *Algebra* (Vol. II., p. 309): "Gauss (*Werke*, Bd. II., p. 525) found, by means of the theory of numbers, two remarkable formulæ of this kind, viz. :—

$$\tfrac{1}{4}\pi = 12\tan^{-1}1/18 + 8\tan^{-1}1/57 - 5\tan^{-1}1/239$$

$$= 12\tan^{-1}1/38 + 20\tan^{-1}1/57 + 7\tan^{-1}1/239 + 24\tan^{-1}1/268 \ \ldots\ (1).$$

By means of this, π could be calculated with great rapidity should its value ever be required beyond the 707th place, which was reached by Mr. Shanks in 1873!"

Mr. Dodgson's formula (C_2) may be thus generalized:

If $\quad \text{A}\cot a = \text{A}\cot(a+x_1) + \text{A}\cot(a+x_2) + \ldots + \text{A}\cot(a+x_n)$,

then $\quad a = ({}_nS_n - {}_nS_{n-2} + {}_nS_{n-4} - \ldots)/({}_nS_{n-1} - {}_nS_{n-3} + {}_nS_{n-5} - \ldots) \ \ldots\ (C_n)$,

where ${}_nS_r$ denotes the sum of the products, r at a time, of the n quantities: $a+x_1$, $a+x_2$, $a+x_3$, ... $a+x_n$. (${}_nS_0$ being taken $= 1$.)

The relation (C_n) may be expanded in various ways. (C_1) reduces to $a = a$, (C_2) reduces to $a^2+1 = xy$ (as already found), (C_3) reduces to $(a^2+1)\{2a+x_1+x_2+x_3\} = x_1x_2x_3$ (C_3), which includes C_2 if we make $x_3 = \infty$, and so on, the disadvantage of the use of any given formula varying with the suffix pertaining to that formula.

From the remarks at the end of the second volume of the *Werke*, it appears that Gauss had obtained a formula, of this class, which fully deserves the epithet "remarkable." We have

$$\begin{aligned} &\text{A}\cot a - \text{A}\cot(2a+x_1) - \text{A}\cot(2a-x_1) \\ &\qquad = \text{A}\cot a - \text{A}\cot\{(4a^2-x_1^2-1)/4a\} \\ &\qquad = \text{A}\cot\left[\{\tfrac{1}{4}(4a^2-x_1^2-1)+1\}/\{\tfrac{1}{4}(4a^2-x_1^2-1)/a-a\}\right] \\ &\qquad = \text{A}\cot\{-a(4a^2-x_1^2+3)/(x_1^2+1)\} = -\text{A}\cot b \text{ say,} \end{aligned}$$

then $b = a\{4(a^2+1)-(x_1^2+1)\}/(x_1^2+1) = 4a(a^2+1)/(x_1^2+1)-a \ldots(2)$;

we have, in fact,

$$\text{A}\cot a + \text{A}\cot\{4a(a^2+1)/(x_1^2+1)-a\} = \text{A}\cot(2a+x_1)+\text{A}\cot(2a-x_1).$$

Hence, in order that b shall be an integer, we must have $4a(a^2+1)/(x_1^2+1)-a$ an integer, that is to say, $4a(a^2+1)/(x_1^2+1)$ must be integral. Thus we give x_1 a value, and then, by means of a table of "prime factors of $aa+1$," we can find suitable values for a, and hence for b.

Gauss's Editor, Schering, to whom we owe the abstraction and preservation of these manuscript remains, gives (*Werke*, Bd. II., p. 525) the following examples of the use of the formulæ (2):—

$a = 253$,	$x_1 = 6$,	$b = 1750507$	$a = 57$,	$x_1 = 3$,	$b = 74043$,
$a = 294$,	$x_1 = 11$,	$b = 832902$	$a = 123$,	$x_1 = 9$,	$b = 90657$.
$a = 119$,	$x_1 = 1$,	$b = 3370437$			

Another relation, which I have found and made use of, can be obtained from the equation $\text{A}\cot a = \text{A}\cot(a-x_1) - \text{A}\cot(a+x_2)$, whence

$$a(x_1+x_2) = a^2+a(x_2-x_1)-x_1x_2+1\,;\quad \therefore\ a^2+1 = x_1(2a+x_2)\ldots\ldots(3).$$

By the aid of the above-given formulæ, but principally of (C_3) and (3), we may very readily "expand" a relation in A cots; thus we have, for $\frac{1}{4}\pi$, the several values:—

$$\begin{gathered} \text{A}\cot 1 = \text{A}\cot 2 + \text{A}\cot 3 = 2\text{A}\cot 3 + \text{A}\cot 7, \\ 2\text{A}\cot 5 + \text{A}\cot 7 + 2\text{A}\cot 8 = 3\text{A}\cot 7 + 2\text{A}\cot 18 + 2\text{A}\cot 8, \\ 5\text{A}\cot 8 + 3\text{A}\cot 57 + 2\text{A}\cot 18 = 5\text{A}\cot 13 + 5\text{A}\cot 21 + 3\text{A}\cot 57 + 2\text{A}\cot 18, \\ 5(\text{A}\cot 18 + \text{A}\cot 47 + \text{A}\cot 21) + 3\text{A}\cot 57 + 2\text{A}\cot 18, \\ 7\text{A}\cot 18 + 5(\text{A}\cot 57 + \text{A}\cot 268 + \text{A}\cot 21) + 3\text{A}\cot 57, \\ 7\text{A}\cot 18 + 8\text{A}\cot 57 + 5(\text{A}\cot 21 + \text{A}\cot 268)\ \ldots\ldots\ldots\ldots(\text{A}). \end{gathered}$$

Now consider $\text{A}\cot 18 - \text{A}\cot 21 - \text{A}\cot 268 - \text{A}\cot 239$,

with relation to the formula

$$(a^2+1)(2a+x_1+x_2+x_3) - x_1x_2x_3 = 0\ \ldots\ldots\ldots\ldots\ldots\ldots(C_3).$$

Here $a = 18$, $x_1 = 3$, $x_2 = 250$, $x_3 = 221$; substituting in (C_3), we get

$$325(36+474) - 3 \times 250 \times 221 = 0 \text{ identically};$$

therefore $\quad A\cot 18 - A\cot 21 - A\cot 268 - A\cot 239 = 0.$

Multiply by 5 and add to (A), when we get

$$\tfrac{1}{4}\pi = 12A\cot 18 + 8A\cot 57 - 5A\cot 239,$$

which is GAUSS I.; in a similar way, but with use of (3), we can easily find $\quad \tfrac{1}{4}\pi = 12A\cot 38 + 20A\cot 57 + 7A\cot 239 + 24A\cot 268,$

which is GAUSS'S II.

6582. (R. A. ROBERTS, M.A.)—If a bicircular quartic meet a conic, show that the sum of the eccentric angles of the eight points of intersection is zero.

Solution by Professor SCHOUTE.

Let $S^2 = 0$ and $Q^4 = 0$ represent the given conic and bicircular quartic. Let $L_i = 0$ ($i = 1, 2, 3, 4$) represent four lines passing through the eight points of intersection of $S^2 = 0$ and $Q^4 = 0$. Then, according to an old theorem, the other eight points common to $Q^4 = 0$ and the four lines $L_i = 0$, not situated on $S^2 = 0$, belong to another conic $T^2 = 0$. So we find

$$Q^4 - \lambda S^2T^2 \equiv \mu L_1L_2L_3L_4 \quad\text{.........................(1).}$$

Now $\quad Q^4 \equiv (x^2+y^2)^2 + \text{\&c.}, \quad S^2 \equiv x^2/p^2 + y^2/q^2 - 1,$

and, putting $L_i \equiv y - m_ix - n_i, \quad T \equiv ax^2 + 2bxy + cy^2 + \text{\&c.},$

the comparison of the coefficients of the terms x^3y and xy^3 in both members of (1) gives

$$p^2\Sigma m_1m_2m_3 = q^2\Sigma m_1 \quad\text{.........................(2).}$$

Now let $L_i = 0$ join the points of $S^2 = 0$, corresponding to the eccentric angles ϕ_i and ψ_i, and put $\phi_i + \psi_i = 2a_i$. Then $m_i = -q/p\cot a_i$. Substituting this result in (2) gives $\Sigma Tg a_1 = \Sigma Tg a_1 Tg a_2 Tg a_3$, or, putting $Tg a_i = z_i$, $\quad (z_1+z_2)(1-z_3z_4) + (z_3+z_4)(1-z_1z_2) = 0,$

i.e. $\quad Tg(a_1+a_2) + Tg(a_3+a_4) = 0$, or $a_1+a_2+a_3+a_4 = (2k+1)\pi$,

i.e. $$\Sigma\phi + \Sigma\psi = 2k'\pi.$$

[The proof is independent of the cubic terms in Q^4. Therefore not only the bicircular quartic, but also the quartics that touch the line l_∞ at infinity in the two cyclic points, satisfy the theorem, which holds true if we substitute "isotropic curve C^{2n}," and $4n$ for "bicircular quartic" (more exactly "isotropic quartic") and eight. So, for $n = 3$, the identity

$$\phi^6 - \lambda S^2T^4 \equiv \mu L_1L_2L_3L_4L_5L_6$$

gives in the same manner (by means of the coefficients of x^5y, x^3y^3, xy^5)

$$p^4\,\Sigma m_1m_2m_3m_4m_5 - p^2q^2\,\Sigma m_1m_2m_3 + q^4\,\Sigma m_1 = 0,$$

which may be transformed into

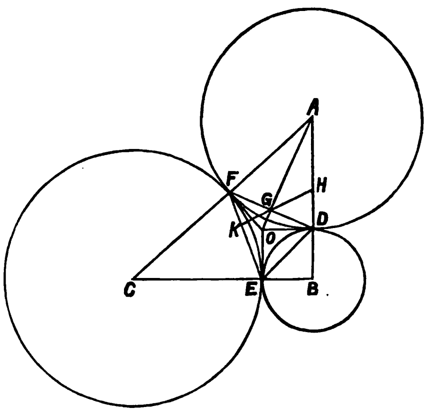
$$Tg(a_1+a_2+a_3)+Tg(a_4+a_5+a_6)=0, \quad i.e. \quad \sum_1^6\phi+\sum_1^6\psi=2k\pi.$$

In the general case, we find

$$Tg(a_1+a_2+\dots a_n)+Tg(a_{n+1}+a_{n+2}+\dots a_{2n})=0, \text{ \&c.}]$$

11809. (J. MACLEOD.)—Three circles whose centres are A, B, C respectively touch in pairs, A and B in the point D; B and C in E; and C and A in F, while ABC is a right angle; DF is bisected in G, and H is taken so that DH : HA = DG : GA. If HG is produced to meet EF in K, prove that HK is perpendicular to EF.

Solution by H. J. CURJEL, B.A.; Professor KOLBE; *and others.*



Let the tangents to the circles at D, E, F meet in O. Then OGA is a straight line, and is at right angles to FD, and GH bisects the angle AGD.

Since OF, OD, OE are equal,

$\angle EFD = \frac{1}{2}\angle EOD$

$= \frac{1}{2}$ a right angle;

but

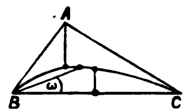
$\angle FGK = \angle HGD$

$= \frac{1}{2}$ a right angle

therefore $\angle FKG$ is a right angle, *i.e.*, HK is perpendicular to EF.

11709. (R. CHARTRES.)—If the base BC of a triangle be the horizontal range of a projectile which passes through the orthocentre and the circumcentre of the triangle: prove (1) that $\cot\omega = 3\cot A$; (2) find the maximum value of A; and show (3) that only with this value of A will it also pass through the Brocard-point.

Solution by the PROPOSER.



Let θ = the angle of projection; then, as in Question 10735, we have

$$\cot C + \cot B = \tan\theta = 2\cot A;$$

$\therefore$ $\cot\omega$ or $\cot A + \cot B + \cot C = 3\cot A$.

The maximum value of $\omega = 30°$; therefore the maximum value of $A = 60°$.

If the projectile also pass through the Brocard-point, we have

$$\tan\omega + \tan(C-\omega) = \tan\theta.$$

Let $x = \cot A$; then, by (1), $(3x^2-1)(216x^6+9x^2+1) = 0$; and the only possible value of this equation is $x^2 = \frac{1}{3}$ or $A = 60°$.

11791. (Professor CATALAN.)—Quelle que soit la base de numération aucun des nombres représentés par 10101, 101010101, 1010101010101,... n'est premier.

Solution by Professors SCHOUTE, KOLBE, *and others*.

Si x représente la base de numération on a

$$\frac{1-x^{4n+2}}{1-x^2} = \frac{1-x^{2n+1}}{1-x}\cdot\frac{1+x^{2n+1}}{1+x},$$

ou $$1+x^2+x^4+\dots x^{4n} = (1+x+x^2+\dots+x^{2n})(1-x+x^2-\dots+x^{2n}).$$

11782. (J. GRIFFITHS, M.A.)—Let the angular points of any triangle ABC be joined with any given point O, and let the joining lines intersect the opposite sides of the triangle in p, q, r; it is required to prove that:—(1) the points p, q, r, together with the middle points of the sides of the triangle and of the segments AO, BO, CO, all lie on the same conic. (2) This conic touches the inscribed and escribed conics of the triangle, which are similar and similarly placed to itself. (3) It passes through the points of intersection, real or imaginary, of the circumscribing and self-conjugate conics of the triangle, which are similar and similarly placed to itself.

Solution by H. W. CURJEL, B.A.

Since the pairs of straight lines, AO, BC; BO, AC; CO, BC may be considered as three conics described through the four points A, B, C, O, they determine an involution on any straight line. Let Ω, Ω' be the double points of the involution determined by these pairs of straight lines on the line at infinity. Then we can project Ω, Ω' orthogonally into the circules. In the projected figure, O becomes the orthocentre of the tri-

angle ABC, and the middle points of any straight lines become the middle points of the corresponding straight lines.

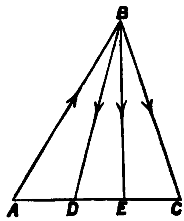
Hence the theorems reduce to (1) the points p, q, r, together with the middle points of the sides of the triangle, and of the segments AO, BO, CO (where O is the orthocentre) all lie on the same conic (the nine-point circle); (2) the nine-point circle touches the inscribed and escribed circles of the triangle; (3) it is coaxal with the circumscribing and self-conjugate circles of the triangle.

11770. (Col. Hime, late R.A.)—Two points, D, E, are taken in the side CA of a triangle ABC such that (n being any number)

$$AD : DC = c^n : a^n, \quad AE : EC = c^{n-2} : a^{n-2};$$

show that the isogonal of the line BD is the isotomic of BE; and hence deduce an easy geometrical construction for the centres of gravity of weights placed at the corners of the triangle proportional to the 2nd, 3rd, ... nth powers of the opposite sides (n being an integer >1).

Solution by the Proposer.



Let the vector (BD) $= \beta$ cut CA so that

$$\frac{AD}{DC} = \frac{c^n}{a^n}.$$

Then $$\frac{c^n}{a^n} = \frac{AD}{DC} = \frac{c\sin\theta}{a\sin(B-\theta)};$$

therefore $$\frac{\sin(B-\theta)}{\sin\theta} = \frac{a^{n-1}}{c^{n-1}}.$$

If (BE) $= \beta'$ be the isogonal of β,

$$\frac{AE}{EC} = \frac{c\sin(B-\theta)}{a\sin\theta} = \frac{a^{n-2}}{c^{n-2}}.$$

If (AB) $= \gamma$ and (BC) $= \alpha$,

$$\beta = \frac{c^n\alpha - a^n\gamma}{c^n + a^n}, \quad \beta' = \frac{a^{n-2}\alpha - c^{n-2}\gamma}{a^{n-2}+c^{n-2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, let (BF) $= \delta$ cut CA so that $\dfrac{AF}{FC} = \dfrac{c^{n-2}}{a^{n-2}}$.

Then, if (BG) $= \delta'$ be the isotomic of δ,

$$\frac{AG}{GC} = \frac{FC}{AF} = \frac{a^{n-2}}{c^{n-2}};$$

therefore $$\delta = \frac{c^{n-2}\alpha - a^{n-2}\gamma}{c^{n-2}+a^{n-2}}, \quad \delta' = \frac{a^{n-2}\alpha - c^{n-2}\gamma}{a^{n-2}+c^{n-2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots 2).$$

Hence, from (1) and (2), $\delta' = \beta'$.

11787. (Professor ORCHARD, M.A., B.Sc.)—In any plane triangle ABC prove that

$$\operatorname{cosec} B \operatorname{cosec} C \sin(B-C)\sin 3A + \operatorname{cosec} A \cot C \sin(C-A)\sin 3B + \operatorname{cosec} A \cot B \sin(A-B)\sin 3C + \operatorname{cosec} A \cot B \sin(C-A)\sin 3B + \operatorname{cosec} A \cot C \sin(A-B)\sin 3C \equiv 0.$$

Solution by Rev. T. R. TERRY, M.A.; W. J. DOBBS, M.A.; *and others.*

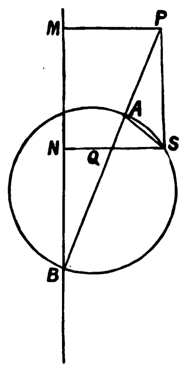
Multiplying throughout by sin B sin C, and combining the second and fourth terms, and the third and fifth terms, it is enough to prove that

$$\sin(B-C)\sin 3A + \sin(C-A)\sin 3B + \sin(A-B)\sin 3C = 0.$$

But the sinister $= (\sin^2 B - \sin^2 C)(3 - 4\sin^2 A) + \ldots + \ldots$, which obviously vanishes.

11736. (Rev. T. ROACH, M.A.)—Given the directrix and two points on an ellipse, find the locus of the focus.

Solution by H. MORGAN BRIERLEY; W. CURJEL, B.A.; *and others.*



Let P, Q be the two given points on the conic, and MN the directrix. Draw PM, QN perpendicular to MN, and divide PQ internally and externally at A and B in the ratio of PM to QN.

On AB as diameter describe a circle SAB, and let S be any point on SAB. Then

$$SP : PM = SQ : QN.$$

Hence the circle SAB is the locus of the foci of conics passing through P and Q and having MN as directrix.

If the conics are ellipses the locus is evidently the part of the arc of the circle between A and the point where it cuts MN and the arc extending the same distance in the opposite direction from A. The circle is the locus of the foci corresponding to the directrix MN only.

11811. (F. G. TAYLOR, M.A., B.Sc.)—Prove that

$$|\cos(\theta_1 - \alpha_1),\ \cos(\theta_2 - \alpha_2),\ \ldots\ \cos(\theta_n - \alpha_n)| = 0.$$

Solution by the PROPOSER; Professor ZERR; *and others.*

Let n lines of length a_1, a_2, &c., forming a closed polygon, make angles α_1, α_2, &c. with a given line.

Project on to a line making an angle θ_1 with the given line. Then

$$a_1 \cos(\theta_1 - \alpha_1) + a_2 \cos(\theta_1 - \alpha_2) + \ldots + a_n \cos(\theta_1 - \alpha_n) = 0.$$

Again project on $n-1$ other lines, and we get $n-1$ similar relations. Eliminating $a_1, a_2, a_3 \ldots a_n$, we get the above result.

[Professor SCHOUTE's solution is as follows:—If $x_1, x_2, \ldots x_n$ satisfy the equations $\sum_1^n x_i \cos \alpha_i = 0$, $\sum_1^n x_i \sin \alpha_i = 0$, the addition of the corresponding elements of the columns multiplied successively by $x_1, x_2, \ldots x_n$ gives a column all the elements of which disappear, &c.]

11678. (ARTEMAS MARTIN, LL.D.)—A wooden hemisphere floats in water, vertex down, with $1\text{-}n^{\text{th}}$ of its axis immersed. Find the specific gravity of the hemisphere.

Solution by H. W. CURJEL, B.A.

The volume immersed $= \dfrac{\pi R^3}{3n^3}(3n-1)$, where R is the radius of the hemisphere; volume of hemisphere $= \frac{2}{3}\pi R^3$; therefore specific gravity

$$= \frac{\dfrac{\pi R^3}{3n^3}(3n-1)}{\frac{2}{3}\pi R^3} = \frac{3n-1}{2n^3}.$$

[Prof. ZERR's solution is as follows:—The volume of water displaced is

$$v = \pi \int_{[(n-1)r]/n}^{r} (r^2 - x^2)\, dx = \frac{\pi r^3}{3n^3}(3n-1);$$

$$\therefore \quad \frac{\pi r^3}{3n^3}(3n-1) = \tfrac{2}{3}\pi r^3 \rho; \quad \text{whence } \rho = \frac{3n-1}{2n^3}.]$$

8617. (R. F. DAVIS, M.A.)—Prove that the semi-axes of the Brocard-ellipse are $R \sin\theta$, $2R \sin^2\theta$, where θ is the Brocard-angle.

Solution by H. J. WOODALL, A.R.C.S.

By geometry of conics, we have product of perpendiculars from the foci on BC is equal to $b_1^2 = (2Rc \sin^2\theta/b) \times (2Rb \sin^2\theta/c) = 4R^2 \sin^4\theta$,

$$a_1^2 = b_1^2 + \tfrac{1}{4}(\text{dist. between foci})^2 = R^2 \sin^2\theta;$$

therefore $\quad a_1 = R \sin\theta$, $b_1 = 2R \sin^2\theta$ (MILNE's *Companion*, p. 108).

9016. (A. GORDON.)—Required the general value in terms of the coefficients of the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n = 0$ of the

determinant $\begin{vmatrix} s_1, & 1, & 0, & 0, & 0, & 0 \\ s_2, & s_1, & 2, & 0, & 0, & \dots\dots \\ s_3, & s_2, & s_1, & 3, & 0, & \dots\dots \\ \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & n-1 \\ s_n, & s_{n-1}, & \dots & \dots & \dots & s_1 \end{vmatrix}$; and express $\Sigma\alpha^3$, β^2 as the sums or determinants and products of determinants, α, β, &c. being the roots of the equation.

Solution by H. J. Woodall, A.R.C.S.

We have, by Newton's equations, and putting the coefficient of $x^n = p_0$,

$$p_0 s_1 + p_1 = 0, \qquad p_0 s_2 + p_1 s_1 + 2p_2 = 0,$$

to
$$p_0 s_n + p_1 s_{n-1} + \dots + p_{n-1} s_1 + np_n = 0.$$

Solve for p_0; therefore

$$p_0 \begin{vmatrix} s_1, & 1, & 0, & 0, & 0 \\ s_2, & s_1, & 2, & 0, & 0 \\ \dots & \dots & \dots & \dots & n-1 \\ s_{n-1}, & s_{n-2}, & \dots & \dots & n-1 \\ s_n, & s_{n-1}, & \dots & \dots & s_1 \end{vmatrix} = -np_n \begin{vmatrix} 0, & 1, & 0, & 0 \\ 0, & s_1, & 2, & 0 \\ \dots & \dots & \dots & n-1 \\ 0, & s_{n-2}, & \dots & n-1 \\ 1, & s_{n-1}, & \dots & s_1 \end{vmatrix}.$$

Now, put $p_0 = 1$; therefore determinant $= (-1)^n p_n n! = \Delta_n$ say, so we have $\Delta_1 = -p_1$, $\Delta_2 = 2p_2$, $\Delta_3 = -6p_3$, and so on.

11692. (J. C. Malet, F.R.S.)—Let the solutions of the equations

$$\frac{d^2y}{dx^2} + 2P_1\frac{dy}{dx} + Q_1 y = 0, \quad \frac{d^2y}{dx^2} + 2P_2\frac{dy}{dx} + Q_2 y = 0 \ \dots\dots\dots \ (a,\ b)$$

be $\qquad y = y_1$ and $y = y_2$; $\quad y = y_3$ and $y = y_4$.

Prove (1) that, if $y_1 y_3 = 1$,

$$M^2 + 2N\left\{(Q_1+Q_2)N - (P_1-P_2)M\right\} = 0 \ \dots\dots\dots\dots\dots \ (1),$$

where
$$M \equiv \frac{dQ_1}{dx} + \frac{dQ_2}{dx} + P_1Q_1 + P_2Q_2 + 3P_1Q_2 + 3P_2Q_1,$$

$$N \equiv \frac{dP_1}{dx} - \frac{dP_2}{dx} + P_1^2 - P_2^2 - Q_1 + Q_2.$$

Hence (2) prove, no relation now being supposed between the solutions of (a) and (b), that the differential equation (non-linear) of which the complete solution is $\quad y = (Ay_1 + By_2)(Cy_3 + Dy_4)$,

where A, B, C, D are arbitrary constants, is $\ V^2 - 2VN(P_1-P_2)\dfrac{dy}{dx}$

$$+ N^2\left\{2y\frac{d^2y}{dx^2} + 2(P_1+P_2)y\frac{dy}{dx} + 2(Q_1+Q_2)y^2 - \frac{dy^2}{dx^2}\right\} = 0 \ \dots\dots \ (2),$$

where
$$V \equiv \frac{d^3y}{dx^3} + 3(P_1+P_2)\frac{d^2y}{dx^2} + L\frac{dy}{dx} + My,$$

$$L \equiv \frac{dP_1}{dx} + \frac{dP_2}{dx} + P_1^2 + P_2^2 - 6P_1P_2 + 2Q_1 + 2Q_2.$$

Hence (3), if $y_1y_4 = y_2y_3$, the linear differential equation, of which the complete solution is $y = C_1y_1y_3 + C_2y_1y_4 + C_3y_2y_4$, where C_1, C_2, C_3 are arbitrary constants, is $V = 0$.

Solution by the PROPOSER.

Substituting for y, in (a) and (b), y_1 and y_3 respectively, and remembering the relation $y_1y_3 = 1$, we easily find, by elimination,

$$2\lambda^2 + 2(P_1 - P_2)\lambda + Q_1 + Q_2 = 0 \quad \ldots\ldots\ldots\ldots (\alpha),$$

where
$$\lambda = \frac{1}{y_1}\cdot\frac{dy_1}{dx}.$$

Differentiating and eliminating $\frac{d^2y_1}{dx^2}$ between the result and

$$\frac{d^2y_1}{dx^2} + 2P_1\frac{dy_1}{dx} + Q_1y_1 = 0,$$

we get
$$2\lambda^2(P_1 + P_2) - 2\lambda\left(\frac{dP_1}{dx} - \frac{dP_2}{dx} - Q_1 + Q_2\right)$$
$$\ldots\ -2P_1Q_2 - 2P_2Q_1 - \frac{dQ_1}{dx} - \frac{dQ_2}{dx} = 0 \quad \ldots\ldots\ldots (\beta).$$

Writing (α) and (β) respectively $a\lambda^2 + b\lambda + c = 0$ and $a'\lambda^2 + b'\lambda + c' = 0$, we find
$$a'c - ac' = 2M, \quad a'b - ab' = 4N,$$
$$bc' - b'c = 2(Q_1 + Q_2)N - 2(P_1 - P_2)M;$$

hence the eliminant of (α) and (β), $(a'b - ab')(bc' - b'c) + (a'c - ac')^2 = 0$, becomes
$$M^2 + 2N\{(Q_1 + Q_2)N - (P_1 - P_2)M\} = 0 \quad \ldots\ldots\ldots (1),$$
which proves the first part of the question.

Supposing now no relation to exist between the solutions of (a) and (b). In (b) change y to yu, when we get the equation

$$\frac{d^2y}{dx^2} + \frac{2}{u}\left(\frac{du}{dx} + P_2u\right)\frac{du}{dx} + \frac{1}{u}\left(\frac{d^2u}{dx^2} + 2P_2\frac{du}{dx} + Q_2u\right)y = 0,$$

of which the solutions are y_3/u and y_4/u.

Now if in M and N we change P_2 to $\frac{1}{u}\cdot\frac{du}{dx} + P_2$, and Q_2 to $\frac{1}{u}\left(\frac{d^2u}{dx^2} + 2P_2\frac{du}{dx} + Q_2u\right)$, N remains unaltered, and M becomes $\frac{1}{u}\left(U - N\frac{du}{dx}\right)$, where

$$U \equiv \frac{d^3u}{dx^3} + 3(P_1 + P_2)\frac{d^2u}{dx^2} + L\frac{du}{dx} + Mu,$$
$$L \equiv \frac{dP_1}{dx} + \frac{dP_2}{dx} + P_1^2 + P_2^2 - 6P_1P_2 + 2Q_1 + 2Q_2.$$

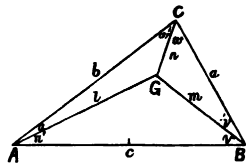
Making the above substitutions in (1), changing u to y, and reducing, we find for the equation (non-linear) of which the complete solution is $y = (Ay_1 + By_2)(Cy_3 + Dy_4)$, is

$$V^2 - 2VN(P_1 - P_2)\frac{dy}{dx} + N^2\left\{2y\frac{d^2y}{dx^2} + 2(P_1 + P_2)y\frac{dy}{dx} + 2(Q_1 + Q_2)y^2 - \frac{dy^2}{dx^2}\right\}$$
$$= 0 \quad \ldots\ldots\ldots\ldots (2).$$

If $y_1y_3 = y_2y_4$, N vanishes, and the linear differential equation of which (in this case) the solution is $y = c_1y_1y_3 + c_2y_1y_4 + c_3y_2y_4$, is $V = 0$.

11682. (Professor HAUGHTON, F.R.S.)—Let there be three chemical atoms α, β, γ, placed at the angles of a certain triangle ABC, and let λ, μ, ν be the coefficients of attraction between β, γ; γ, α; α, β. If the triangle revolve in steady motion, in its own plane, round the common centre of gravity of α, β, γ, prove (1) that the species of the triangle is given by the proportions $a^3 : b^3 : c^3 = \lambda, \mu, \nu$; and find (2) the other conditions of steady motion.

Solution by the PROPOSER.



Let the accompanying figure show the general construction, where a, b, c are the sides of the triangle, l, m, n are the lines drawn from the angles to the centre of gravity; and let these lines divide the angle A into u and u', the angle B into v and v', and the angle C into w and w'. Each atom is attracted by the other two, and must fulfil the mechanical conditions that (1) the tangential components must be equal; and (2) the radial components must equal the centrifugal force.

From the first and second conditions, we find, respectively,

$$\frac{\nu}{\mu}\,\frac{\beta}{\gamma}\,\frac{b^2}{c^2} = \frac{\sin u}{\sin u'},\quad \frac{\lambda}{\nu}\,\frac{\gamma}{\alpha}\,\frac{c^2}{a^2} = \frac{\sin v}{\sin v'},\quad \frac{\mu}{\lambda}\,\frac{\alpha}{\beta}\,\frac{a^2}{b^2} = \frac{\sin w}{\sin w'}\;\ldots(a,\,b,\,c),$$

$$\frac{\mu\gamma}{b^2}\cos u + \frac{\nu\beta}{c^2}\cos u' = \Omega^2 l,\qquad \frac{\nu\alpha}{c^2}\cos v + \frac{\lambda\gamma}{a^2}\cos v' = \Omega^2 m\;\ldots\ldots(d,\,e),$$

$$\frac{\lambda\beta}{a^2}\cos w + \frac{\mu\alpha}{b^2}\cos w' = \Omega^2 n\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(f),$$

Ω denoting the common rotation of all three.

We have also, from the geometrical properties of the centre of gravity,

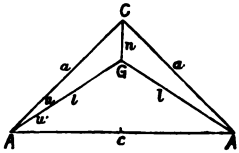
$$\frac{\sin u}{\sin C} = \frac{a}{l}\,\frac{\beta}{\alpha+\beta+\gamma},\quad \frac{\sin v}{\sin A} = \frac{b}{m}\,\frac{\gamma}{\alpha+\beta+\gamma},\quad \frac{\sin w}{\sin B} = \frac{c}{n}\,\frac{\alpha}{\alpha+\beta+\gamma}\ldots(g,h,i),$$

$$\frac{\sin u'}{\sin B} = \frac{a}{l}\,\frac{\gamma}{\alpha+\beta+\gamma},\quad \frac{\sin v'}{\sin C} = \frac{b}{m}\,\frac{\alpha}{\alpha+\beta+\gamma},\quad \frac{\sin w'}{\sin A} = \frac{c}{n}\,\frac{\beta}{\alpha+\beta+\gamma}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(g',\,h',\,i').$$

From this we find $\quad \dfrac{\nu}{\mu}\,\dfrac{b^2}{c^2} = \dfrac{\sin C}{\sin B}$, &c., &c.;

also, $\quad \lambda : \mu : \nu = a^3 : b^3 : c^3.$



The Laplacian triangle becomes isosceles in the case of water, the oxygen atom being placed at the vertex C, and the hydrogen atoms at the extremities of the base A, A. If n be made equal to CG, where G is the common centre of gravity, it is easy to see, since GA is unity,

$a^2 = 17n^2 + 1$, $\sin A = 9na^{-1}$, $\sin u' = 8n$.

From the dynamical principles already laid down, we find

$$\frac{\mu}{c^2}\cos u' + \frac{\mu''\beta}{c^2}\cos u = \omega''^2, \qquad \frac{\mu}{c^2}\sin u' = \frac{\mu''\beta}{c^2}\cos u \quad \ldots\ldots(1,\ 2),$$

where μ is the coefficient of attraction between hydrogen and hydrogen, μ'' the coefficients of attraction between oxygen and hydrogen in the combination H_2O, and β the atomic weight of oxygen.

Eliminating μ/c^2 between the two equations, we find

$$\omega''^2 = \frac{\mu''\beta}{a^2}\,\frac{\sin A}{\sin u'}, \quad \text{or} \quad \omega''^2 = \frac{9}{8}\,\frac{\mu''\beta}{(17n^2+1)^{\frac{3}{2}}}.$$

Hence, for a given coefficient of attraction, ω''^2 (which measures the stability of the molecule) will be a maximum when $n^2 = 0$; or, in other words, the dumb-bell form of molecule, when the vertex of the triangle falls upon the base (or when the two sides of the triangle become equal to, or less than, the third), is the most economical configuration for producing a given stability of molecule.

The condition for a real triangle is, of course,

$$2\mu''^{\frac{1}{2}} > \mu^{\frac{1}{2}}, \quad \text{or} \quad \mu'' > \tfrac{1}{2}, \quad \text{or} \quad \mu''\beta > 8.$$

When $\mu''\beta = 8$ the triangle begins to degenerate into the dumb-bell, and

$$\omega''^2 = \tfrac{9}{8}(\mu''\beta) = 9.$$

This amount of stability is greater than we want, and therefore the sum of the two sides of the Laplacian triangle is less than the base.

The coefficient of attraction between oxygen and hydrogen in water (H_2O) is different from the coefficient in hydroxyl (HO).

The action of nitrogen upon hydrogen in ammonia is *repulsive*, varying as the inverse square of distance, and the action of carbon upon hydrogen in marsh gas is also repulsive. I believe that the announcement of repulsive forces in chemistry will be welcome to students both of physics and of biology.

11795. (Professor Macfarlane.)—Prove that

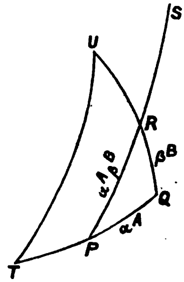
$$\cos nA \cos nB = (\cos A \cos B)^n$$
$$+\frac{n(n-1)}{1\,.\,2}(\cos A \cos B)^{n-2}(1-\cos^2 A-\cos^2 B+3\cos^2 A\cos^2 B)$$
$$+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}(\cos A \cos B)^{n-4}(1-\cos^2 A\cos^2 B)^2.$$

Solution by the Proposer; Prof. Zerr, M.A.; *and others.*

The equation follows from a law of indices which many writers hold to be one of the fundamental laws of algebra; namely that $(ab)^n = a^n b^n$.

But the equation is not true; hence the law in question is true only under certain conditions.

Let α denote the axis of a great circle on a unit sphere, and A an amount of its arc; β the axis of another great circle, and B an amount of its arc. These are represented on the diagram by PQ and QR; and PR, the arc of the great circle from P to R, represents the product $\alpha^A\beta^B$.



Now $\qquad \alpha^A = \cos A + \sin A \,.\, \alpha^{\frac{1}{2}\pi}$,

and $\qquad \beta^B = \cos B + \sin B \,.\, \beta^{\frac{1}{2}\pi}$;

$$\begin{aligned}\therefore\ \alpha^A\beta^B &= (\cos A + \sin A \,.\, \alpha^{\frac{1}{2}\pi})(\cos B + \sin B \,.\, \beta^{\frac{1}{2}\pi}.)\\ &= (\cos A \cos B - \sin A \sin B \cos \alpha\beta)\\ &\quad + \{\cos B \sin A \,.\, \alpha + \cos A \sin B \,.\, \beta\\ &\qquad - \sin A \sin B \sin \alpha\beta \,.\, (\alpha\beta)^{\frac{1}{2}\pi}\}.\end{aligned}$$

Let C denote the arc, and γ the axis of the product angle; then

$$\cos C = \cos A \cos B - \sin A \sin B \cos \alpha\beta,$$

and $\sin C \,.\, \gamma = \cos B \sin A \,.\, \alpha + \cos A \sin B \,.\, \beta - \sin A \sin B \sin \alpha\beta \,.\, (\alpha\beta)$.

$$\begin{aligned}\text{Now}\quad (\alpha^A\beta^B)_n &= (\cos C + \sin C \,.\, \gamma^{\frac{1}{2}\pi})^n\\ &= \cos^n C + n \cos^{n-1} C \sin C \,.\, \gamma^{\frac{1}{2}\pi} + \frac{n(n-1)}{2!}\cos^{n-2} C \sin^2 C \,.\, \gamma^{\pi} + \dots .\end{aligned}$$

$$\text{Hence}\quad \cos(\alpha^A\beta^B)^n = \cos^n C - \frac{n(n-1)}{2!}\cos^{n-2} C \sin^2 C + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4} C \sin^4 C,$$

$$\text{and}\quad \sin(\alpha^A\beta^B)^n = n\cos^{n-1} C \sin C - \frac{n(n-1)(n-2)}{3!}\cos^{n-3} C \sin^3 C + \dots .$$

Now by $(\alpha^A\beta^B)^n$ is meant n times the arc PR; for instance, PS represents $(\alpha^A\beta^B)^2$. But by $\alpha^{nA}\beta^{nB}$ is meant n times the arc PQ followed by n times the arc QR; for instance, TU represents $\alpha^{2A}\beta^{2B}$. But PS and TU have different axes and unequal arcs; hence, when the axes α and β are different, $(\alpha^A\beta^B)^n$ is not $= \alpha^{nA}\beta^{nB}$. Suppose that the arcs are equal in magnitude though the axes are not the same, then it would be true that

$$\cos nA \cos nB - \sin nA \sin nB \cos \alpha\beta = \cos^n C - \frac{n(n-1)}{2!}\cos^{n-2} C \sin^2 C + \&c.$$

This is the erroneous assumption which I made.

As a particular case, let β be perpendicular to α; then

$$\cos C = \cos A \cos B,$$

and $\qquad \sin C \,.\, \gamma = \cos B \sin A \,.\, \alpha + \cos A \sin B \,.\, \beta - \sin A \sin B \,.\, (\alpha\beta)$,

where the three components are mutually at right angles; therefore

$$\sin^2 C = \cos^2 B \sin^2 A + \cos^2 A \sin^2 B + \sin^2 A \sin^2 B = 1 - \cos^2 A \cos^2 B.$$

Under the above condition

$$\cos(\alpha^{A}\beta^{B})^{n} = (\cos A\cos B)^{n} - \frac{n(n-1)}{2!}(\cos A\cos B)^{n-2}(1-\cos^2 A\cos^2 B)$$
$$+\frac{n(n-1)(n-2)(n-3)}{4!}(\cos A\cos B)^{n-4}(1-\cos^2 A\cos^2 B)^2, \text{ \&c.};$$

but the above series is not equal to $\cos nA \cos nB$, as the axes are different.

What is new in notation and in principle in the above investigation is fully explained in my memoir on "The Principles of the Algebra of Physics," and subsequent papers "On the Imaginary of Algebra," and "The Fundamental Theorems of Analysis Generalized for Space."

11667. (ARCHDEACON WILSON, M.A.)—When $4n+1$ is a prime number, it is an old and well-known property of numbers that it is expressible in the form of two squares. But the proofs throw little or no light on "the reason why." Can any connexion be shown, or any explanation be given of this curious property?

Solution by MORGAN BRIERLEY.

PETER BARLOW, in his *Theory of Numbers*, 1811, gives a somewhat recondite, but perfectly satisfactory demonstration of the curious property here mentioned, derived from first principles, but, if it can be established by induction, something of "the reason why" may be seen, as follows:—

Let $4n+1$, $4n+1$, $4n+1$, $4n+1$, $4n+1$ be an arithmetical series of odd numbers, in which the common difference of the terms is 4, and let n be successively interpreted by the digits in the natural series 0, 1, 2, 3, 4, 5, &c., whence we get 1, 5, 9, 13, 17, 21, 25, 29, &c.

Let now the missing terms of the natural series be interpolated, and we get 0, 1; 2, 3, 4, 5; 6, 7, 8, 9; 10, 11, 12, 13; 14, 15, 16, 17; &c., in which it will be seen that the last term, $4n+1$, is the sum of any two terms equi-distant from each end of the series; *i.e.*, $1+16 = 17$, $4+13 = 17$, &c. As is known, if the odd numbers be successively added, there results a series of squares 1, 4, 9, 16, 25, 36, &c., each of which may be placed under its root; thus

1, 3, 5, 7, 9, 11, 13, &c.;
1, 4, 9, 16, 25, 36, 49, &c.

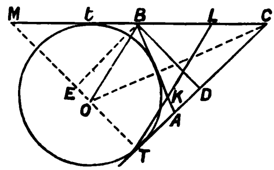
Take now, any of the terms $4n+1$, in the natural series, and it will be seen that a square number is at an equal distance, necessarily, from each end of the series; thus, $21 \neq 4n+1$, which, however, is not a prime, and the squares, 4 and 16, occupy respectively the fifth place from the origin and from the end of the series. This, obviously, will hold of every series of which the last term is of the form $4n+1$; consequently, when it is a prime, it is necessarily the sum of two squares. For example, $5 = 1+4$, 1 and 4 being equi-distant from 0 and 5; $13 = 4+9$, 4 and 9 being respectively the fifth term from each end of the series.

456. (J. H. Swale.)—In any plane triangle ACB draw BD perpendicular to the opposite side AC, and let T be the point of contact of the inscribed circle with AC; draw TKL parallel to the line bisecting the angle B, and meeting BC, BA at K and L; then prove that (1) BK = BL = DT, and TK . TL = DB²; and (2) the same is also true for each side and its opposite angle.

455. (J. H. Swale.)—Let CT, C*t* be tangents to a circle; from T, one of the points of contact, and O, the centre of the circle, draw any parallels TL, OB to the other tangent at L, B; and to TL apply BK = BL; then prove that BK will be a tangent, and TL . TK = BD², BD being a perpendicular on CT.

Solution by J. C. St. Clair.

These two questions are different forms of the same theorem. The following is a solution of Question 456.



We have

angle *t*BO = BLK = BKL = OBK;

∴ BK is a tangent to circle.

Produce TO to meet C*t* in M, and from B draw BE parallel to CT. Join OC. Then MO : OT = MB : BL.

Also MC : CT = MB : BE.

And, since CO bisects the vertical angle of triangle MCT,

$$\text{MO} : \text{OT} = \text{MC} : \text{CT}; \quad \therefore \ \text{BE} = \text{BL},$$

and a circle, centre B, and radius BL = BK, will touch TM at E.

Hence $$\text{TK} . \text{TL} = \text{TE}^2 = \text{BD}^2$$

11833. (Professor Catalan.) *a* étant une constante positive, démontrer que

$$u \equiv \int_0^1 \frac{x^a(1+x-2x^{a+1})}{1-x^2}\,dx = \log_e 2.$$

Solution by Professor Sebastian Sircom.

$$\frac{du}{da} = \int_0^1 \frac{x^a \log x}{1-x}\,dx - \int_0^1 \frac{(x^2)^a \log(x^2)}{1-x^2}\,d(x^2) = 0, \text{ if } a \text{ is positive,}$$

since the integrals are finite and equal; hence *u* is independent of *a*, and putting $a = 0$, we have $u = \int_0^1 \frac{1-x}{1-x^2}\,dx = \int_0^1 \frac{dx}{1+x} = \log_e 2,$

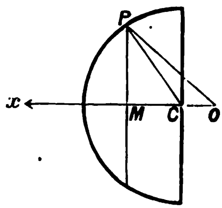
11675. (J. W. Russell, M.A.)—A particle is placed at O on the axis of a solid homogeneous hemisphere whose centre is C, very near to C and outside the solid. Show that the difference between the attraction of the hemisphere on the particle at O and on the particle when placed at C is equal to the attraction of the completed solid sphere on the particle at O.

Solution by the Proposer.

If OC $= z$, then the attraction at O of

$$\text{hemisphere} = \int 2\pi\rho dx\,(1-\cos \text{POM})$$

$$= \int 2\pi\rho dx\left(1-\frac{x+z}{\text{OP}}\right).$$



Hence the difference of attractions is

$$= \int_0^{\frac{1}{2}\pi} 2\pi\rho a\cos\theta d\theta \times\left[\frac{a\sin\theta+z}{\left[(a\sin\theta+z)^2+a^2\cos^2\theta\right]^{\frac{1}{2}}}-\sin\theta\right]$$

where $x = a\sin\theta,\quad y = a\cos\theta$

$$= \int_0^{\frac{1}{2}\pi} 2\pi\rho a\cos\theta\, d\theta\,\frac{(a\sin\theta+z-\text{R}\sin\theta)}{\text{R}}$$

where $\text{R} = \left[(a\sin\theta+z)^2+a^2\cos^2\theta\right]^{\frac{1}{2}}$

$$= \int_0^{\frac{1}{2}\pi} 2\pi\rho a\cos\theta\, d\theta\,\frac{(a^2\sin^2\theta+2az\sin\theta+z^2-a^2\sin^2\theta-2az\sin^3\theta-z^2\sin^2\theta)}{\text{R}\,(a\sin\theta+z+\text{R}\sin\theta)}$$

$$= \int_0^{\frac{1}{2}\pi} 2\pi\rho a\cos\theta\, d\theta\,\frac{2z\sin\theta\cos^2\theta}{2a\sin\theta}\quad(\text{to first order of } z)$$

$$= \int_0^{\frac{1}{2}\pi} z\,2\pi\rho\cos^3\theta\, d\theta = 2\pi\rho z\tfrac{2}{3} = \tfrac{4}{3}\pi\rho z.$$

[Prof. Dewar states that this is a case of a problem given in Routh's *Analytical Statics*, Vol. ii., p. 43, Ex. 9, which was proposed at St. John's College in 1886. He would like the Proposer to favour us with the solution of the more general question, where x, y, z are the coordinates of the small distances (CO), where O may be either inside or outside.]

11827. (Professor de Longchamps.) — On considère une hyperbole équilatère H et le quadrilatère formé par les tangentes aux points d'incidence des normales issues d'un même point. Démontrer que les circonférences décrites sur les diagonales du quadrilatère comme diamètres passent par le centre de H, et, en ce point, sont mutuellement tangentes.

Solutions by W. J. Greenstreet, M.A.; W. J. Dobbs, B.A.; *and others.*

If the four normals through P (α, β) meet the curve in A, B, C, D, we know that A, B, C, D lie on another equilateral hyperbola, and that any one of these four points is the orthocentre of the remaining three.

Hence, if L and M be the poles of AB and CD, two lines at right angles, OL, OM are also at right angles (O is the centre of the hyperbola); and therefore the circle on MN as diameter passes through O.

Also, by NEWTON'S Theorem (*Cremona*, § 318), the join of the mid-points of the diagonals of ABCD passes through O, whence the rest follows.

Analytically, we get $2xy - x\beta - y\alpha = 0$ as the curve upon which lie A, B, C, D, the feet of the normals concurrent at (α, β); the original hyperbola being $x^2 - y^2 = a^2$. If the poles be L and M, (x_1, y_1 and x_2, y_2), then the polars are given by $(xx_1 - yy_1 - a^{-2})(xx_2 - yy_2 - a^2) = 0$.

They must also be identical with AB and CD, the common chords AB, CD of the two curves, which are given by

$$\lambda(x^2 - y^2 - a^2) + \mu(2xy - x\beta - y\alpha) = 0.$$

This gives us $x_1x_2 + y_1y_2 = \lambda - \lambda = 0$; $\beta/\alpha = -(x_1 + x_2)/(y_1 + y_2)$, the circle on LM as diameter being

$$[x - (x_1 + x_2)/2]^2 + [y - (y_1 + y_2)/2]^2 = (x_1 - x_2)^2/4 + (y_1 - y_2)^2/4;$$

this circle passes through O, because $x_1x_2 + y_1y_2 = 0$.

The tangent to this circle at O makes an angle

$$\tan^{-1}[-(x_1 + x_2)/(y_1 + y_2)] = \tan^{-1}\beta/\alpha$$

with the axis of λ, and therefore is OP. Therefore, &c.

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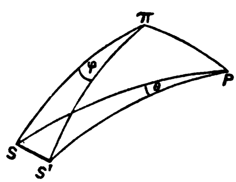
631. (M. COLLINS, B.A.)—Required the locus of all stars that have the same precession in right ascension.

Solution by W. J. GREENSTREET, M.A.



Let P, Π be the poles of the equator and ecliptic respectively, S′ the precessional displacement of S relative to P in any given time; then $\quad S\Pi S' = \phi = \text{constant}.$

Let $SPS' = \theta$; then

$$\theta = (\phi/\sin S\Pi)\sin SP \,.\, \cos \Pi SP$$

and $\quad \cos \Pi SP = \dfrac{\cos\omega - \cos SP \,.\, \cos S\Pi)}{(\sin SP \,.\, \sin S\Pi)},$

where ω is the obliquity of the ecliptic;

$$\therefore \quad \theta = \phi(\cos\omega - \cos SP \,.\, \cos S\Pi)/\sin^2 S\Pi.$$

Hence, taking r the radius of the celestial sphere, the line of the equinoxes as the axis of y, and the earth's axis as that of z, we have

$$r\cos SP = z; \quad r\cos S\Pi = z\cos\omega + x\sin\omega; \quad r^2\sin SP = x^2 + y^2;$$

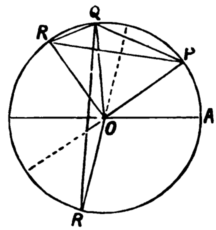
$$\therefore \quad (x^2 + y^2)\,\theta = \phi\,[\cos\omega\,\Sigma x^2 - z(z\cos\omega + x\sin\omega)]:$$

$$i.e., \quad (x^2 + y^2)(\phi\cos\omega - \theta) = \phi\sin\omega\, xz,$$

a cone of the second degree, one of its axes the axis of y, *i.e.*, the line of the equinoxes.

11760 and **11764.** (Professors MUKHOPADHYAY and BHATTHACHARYA).—Prove that the mean value (1) of the area of all the acute-angled triangles inscribed in a given circle of radius a is $3a^2/\pi$; (2) of all the obtuse-angled triangles is a^2/π; (3) of the perimeter of all the acute-angled triangles inscribed in a given circle of radius a is $48a/\pi^2$; and (4) of all the obtuse-angled triangles is $16(\pi-1)a/\pi^2$.

Solution by Professor ZERR; H. W. CURJEL, M.A.; *and others.*



Let $OA = a$, $\angle AOR = \theta$, $\angle POA = \phi$, $\angle QOA = \psi$. Then

$$\tfrac{1}{2}a^2\left[\sin(\theta-\psi)+\sin(\psi-\phi)+\sin(\phi-\theta)\right] = u = \text{area};$$

$$2a\left[\sin\tfrac{1}{2}(\theta-\psi)+\sin\tfrac{1}{2}(\psi-\phi)+\sin\tfrac{1}{2}(\theta-\phi)\right] = p = \text{perimeter}.$$

For the acute triangle, the limits of θ are π and 2π; of ϕ, 0 and $\theta-\pi$; of ψ, $\theta-\pi$ and $\phi+\pi$.

For the obtuse triangle, the limits of θ are 0 and 2π; of ϕ, $\theta-\pi$ and θ; of ψ, ϕ and θ.

$$\therefore\quad \Delta = \left(\int_\pi^{2\pi}\int_0^{\theta-\pi}\int_{\theta-\pi}^{\phi+\pi} u\,ad\theta\,ad\phi\,ad\psi\right)\Big/\left(\int_\pi^{2\pi}\int_0^{\theta-\pi}\int_{\theta-\pi}^{\phi+\pi} ad\theta\,ad\phi\,ad\psi\right)$$

$$= \frac{3a^2}{2\pi^3}\int_\pi^{2\pi}\int_0^{\theta-\pi}\left[2-2\cos(\theta-\phi)+(2\pi+\phi-\theta)\sin(\phi-\theta)\right]d\theta\,d\phi = 3\frac{a^2}{\pi};$$

$$\Delta_1 = \left(\int_0^{2\pi}\int_{\theta-\pi}^{\theta}\int_\phi^\theta u\,ad\theta\,ad\phi\,ad\psi\right)\Big/\left(\int_0^{2\pi}\int_{\theta-\pi}^{\theta}\int_\phi^\theta ad\theta\,ad\phi\,ad\psi\right)$$

$$= \frac{a^2}{2\pi^3}\int_0^{2\pi}\int_{\theta-\pi}^{\theta}\left[(\theta-\phi)\sin(\phi-\theta)+2-2\cos(\theta-\phi)\right]d\theta\,d\phi = \frac{a^2}{\pi};$$

$$L = \left(\int_\pi^{2\pi}\int_0^{\theta-\pi}\int_{\theta-\pi}^{\phi+\pi} p\,ad\theta\,ad\phi\,ad\psi\right)\Big/\left(\int_\pi^{2\pi}\int_0^{\theta-\pi}\int_{\theta-\pi}^{\phi+\pi} ad\theta\,ad\phi\,ad\psi\right)$$

$$= \frac{6a}{\pi^3}\int_\pi^{2\pi}\int_0^{\theta-\pi}(4+2\pi+\phi-\theta)\sin\tfrac{1}{2}(\theta-\phi)\,d\theta\,d\phi = 48a/\pi^2;$$

$$L_1 = \left(\int_0^{2\pi}\int_{\theta-\pi}^{\theta}\int_\phi^\theta p\,ad\theta\,ad\phi\,ad\psi\right)\Big/\left(\int_0^{2\pi}\int_{\theta-\pi}^{\theta}\int_\phi^\theta ad\theta\,ad\phi\,ad\psi\right)$$

$$= \frac{2a}{\pi^3}\int_0^{2\pi}\int_{\theta-\pi}^{\theta}\left[4-4\cos\tfrac{1}{2}(\theta-\phi)+(\theta-\phi)\sin\tfrac{1}{2}(\theta-\phi)\right]d\theta\,d\phi = 16(\pi-1)\frac{a^2}{\pi^2}.$$

11409. (Professor MINCHIN, M.A.)—A straight cylindrical wire has a line marked on its surface parallel to its axis. It is then laid along the surface of a right cone (semi-vertical angle α) so that the marked line cuts the generators everywhere at a constant angle (i). Prove that the rate of twist at any point of the wire is $(\sin i\cos i\cos\alpha)/r$, where r is the distance of the point from the axis of the cone.

Solution by H. W. Curjel, B.A.

The rate of twist is evidently the rate of torsion $(1/\sigma)$ of the curve o contact.

Taking the axis of the cone as the axis of y, the vertex being at the origin, the equations to the curve of contact may be written

$$x = r\cos\theta,\ \ z = r\sin\theta,\ \ y = s\cos\alpha\cos i,\ \ \theta = \frac{\tan i}{\sin\alpha}\log s,\ \ s = \frac{r}{\sin\alpha\cos i};$$

$$\therefore\ \frac{dx}{ds} = \sin\alpha\cos i\sin\theta + \sin i\cos\theta,\qquad \frac{dz}{ds} = \sin\alpha\cos i\cos\theta - \sin i\sin\theta,$$

$$\frac{dy}{ds} = \cos\alpha\cos i,\qquad \frac{d^2x}{ds^2} = \frac{\sin i}{s}\left(\cos\theta - \sin\theta\frac{\tan i}{\sin\alpha}\right),$$

$$\frac{d^2z}{ds^2} = -\frac{\sin i}{s}\left(\sin\theta + \cos\theta\frac{\tan i}{\sin\alpha}\right),\qquad \frac{d^2y}{ds^2} = 0,$$

$$\frac{d^3x}{ds^3} = -\frac{\sin i\cos\theta}{s^2}\left(1 + \frac{\tan^2 i}{\sin^2\alpha}\right),\qquad \frac{d^3z}{ds^3} = \frac{\sin i\sin\theta}{s^2}\left(1 + \frac{\tan^2 i}{\sin^2\alpha}\right).$$

$$\text{Then}\quad \frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 = \frac{\sin^2 i}{s^2}\left(1 + \frac{\tan^2 i}{\sin^2\alpha}\right),$$

where ρ = radius of curvature,

$$\text{and}\qquad \frac{1}{\rho^2\sigma} = \begin{vmatrix} \frac{dx}{ds}, & \frac{dy}{ds}, & \frac{dz}{ds} \\ \frac{d^2x}{ds^2}, & \frac{d^2y}{ds^2}, & \frac{d^2z}{ds^2} \\ \frac{d^3x}{ds^3}, & \frac{d^3y}{ds^3}, & \frac{d^3z}{ds^3} \end{vmatrix}$$

$$= \frac{\sin^2 i}{s^3}\left(1 + \frac{\tan^2 i}{\sin^2\alpha}\right)\cos\alpha\cos i \begin{vmatrix} -\sin\theta - \cos\theta\frac{\tan i}{\sin\alpha}, & \cos\theta - \sin\theta\frac{\tan i}{\sin\alpha} \\ \sin\theta, & -\cos\theta \end{vmatrix};$$

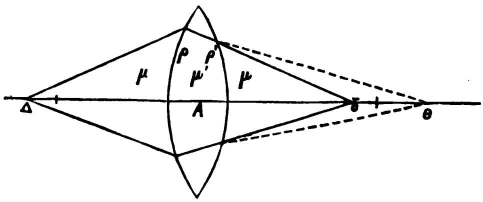
$$\therefore\ \frac{1}{\sigma} = \frac{\cos\alpha\cos i}{s}\,\frac{\tan i}{\sin\alpha} = \frac{\cos\alpha\sin i}{s\sin\alpha}$$

$$= \frac{\sin\alpha\cos i}{r}\,\frac{\cos\alpha\sin i}{\sin\alpha} = \frac{\sin i\cos i\cos\alpha}{r}.$$

3517. (Rev. T. Mitcheson, B.A.)—If β, γ be the distances of the conjugate foci from the centre of a double convex lens, whose thickness may be disregarded, for a ray of light diverging from Δ and converging to δ on the other side of the lens; ρ, ρ_1 the radii of the spherical surfaces; and α the distance of the focus to which the ray would converge, were the medium after the first refraction of uniform density; prove that

$$\alpha = \frac{\beta(\gamma+\rho_1)\rho + \gamma(\beta+\rho)\rho_1}{\beta(\gamma+\rho_1) - \gamma(\beta+\rho)}.$$

Solution by H. J. WOODALL, A.R.C.S.



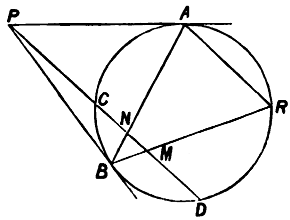
By the theory of geometrical optics, we have

$$\mu'\beta/(\beta+\rho) = -\mu\alpha/(\rho-\alpha) \quad \text{and} \quad \mu'\gamma/(\rho'+\gamma) = \mu\alpha/(\rho'+\alpha);$$

from these, eliminating $\mu' : \mu$, we get the expression for α as given n the question.

11836. (EDITOR.)—From a point P there are drawn, to a circle, two tangents PA, PB, and a chord PCD; prove that, (1) if a chord AR be drawn parallel to PD, the chord BR will bisect CD; and (2) that the theorem is true for any conic.

Solution by J. C. ST. CLAIR, M. BRIERLEY, *and others.*



(1) Since (R . CBDA) = (B . CBDA);

we have, on the transversal PD,

$$(\text{CMD}\,\infty) = (\text{CPDN}).$$

But the latter ratio is harmonic, and therefore CD is bisected in M.

(2) In like manner may the theorem be proved to be true for any conic section.

11206. (Professor MADHAVARO.)—A pack of cards, equal or unequal, stands on the edge of a horizontal table, each card projecting beyond the one just below it. If the highest card project as far as possible from the table, show that each card is on the point of moving independently of the rest.

Solution by H. W. CURJEL, B.A.

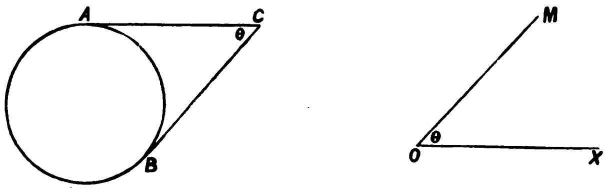
If the pack be supposed to become firm, except between two adjacent cards, if the upper pack is not on the point of moving, it may be moved forward; but as this moves the centre of gravity of the whole pack

forward, the whole pack must be moved back to be in equilibrium. But the centre of gravity of the whole pack evidently is not moved forward by moving the upper pack as much as the upper pack. Therefore the top card will project more by moving the upper pack as far forward as possible, and then moving the whole pack back so as to be in equilibrium. Therefore, when the highest card projects as far as possible, each card is on the point of moving.

11177. (H. Brocard.) — Les tangentes menées en un point fixe A et en un point variable B d'une circonférence Δ se rencontrent en un point C. Par un point fixe O, on mène une droite OM égale et parallèle à BC. Démontrer que le point M décrit une strophoïde droite (logocyclique).

Solution by H. W. Curjel, B.A.

Take the initial line OX parallel to AC. Let OM $= r$ and $\angle$ MOX $= \angle$ ACB $= \theta$. Let radius of circle $= a$.



$$r = \text{BC} = a \cot \tfrac{1}{2}\theta;$$

therefore the locus is a logocyclic curve.

The Cartesian equation is $\quad (y-a)^2(x^2+y^2) = a^2x^2.$

11637. (R. Tucker, M.A.)—Two tangents OP, OQ to a parabola meet at an angle ω; prove that (1) if $\omega = \cos^{-1}\frac{1}{3}$, then the orthocentre of the triangle OPQ (when OP $=$ OQ) lies on the curve; (2) if the corresponding chord of the evolute subtends a right angle at the focus, then PQ cannot be a focal chord; and (3) if λ, μ be the cotangents of the acute angles made by OP, OQ with the axis, then, generally,

$$4\lambda^3\mu^3 = (1+3\lambda^2)(1+3\mu^2).$$

Solution by R. Knowles, B.A.

Let the coordinates of O, P, Q be $(h, 0)$, (x_1, y_1), (x_2, y_2); then, since O is on the axis, $h = -x_1$, $x_1 = x_2$, $y_2 = -y_1$,

$$\text{OP}^2 = \text{OQ}^2 = 4h(h-a), \quad \text{and} \quad \text{PQ}^2 = 4y_1^2 = -16ah;$$

from $\triangle OPQ$, $\cos\omega = (OP^2+OQ^2-PQ^2)/2OP\,.\,OQ = (h+a)/(h-a)$;
whence $h = a(\cos\omega+1)/(\cos\omega-1) = -2a$, when $\cos\omega = \frac{1}{3}$;
and in this case $x_1 = x_2 = 2a$, $y_1 = 2a\sqrt{2}$, $y_2 = -2a\sqrt{2}$;
the equations to the perpendiculars from P on OQ and from Q on OP are

$$y-2a\sqrt{2} = \sqrt{2}\,(x-2a), \quad y+2a\sqrt{2} = -\sqrt{2}\,(x-2a),$$

and these intersect in the vertex, which proves (1).

(2) The corresponding chord on the evolute passes through the points

$$(3x_1+2a), \quad -y_1^3/4a^2; \quad (3x_2+2a), \quad -y_2^3/4a^2;$$

and the condition that it should subtend a right angle at the focus is

$$y_1^3y_2^3/16a^4(3x_1+a)(3x_2+a)+1 = 0 \quad \dots\dots\dots\dots\dots\dots \quad (\alpha),$$

or

$$y_1^3 = 4a^2(3x_1+a),$$

which becomes

$$(1+\cos\omega)^{\frac{3}{2}}/32a^2(1-\cos\omega)^{\frac{3}{2}} = -3a(\cos\omega+1)/(\cos\omega-1)+a;$$

and, as this equation is not satisfied by $\cos\omega = 0$, PQ cannot be a focal chord.

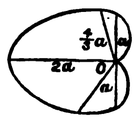
(3) Substituting in equation (α), which may be written

$$y_1^3y_2^3 = -a^2(4a^2+3y_1^2)(4a^2+3y_2^2), \quad y_1 = 2a\lambda, \quad y_2 = -2a\mu,$$

we have the stated result.

11695. (Professor SVECHNICOFF.)—Quelle est, parmi les normales à une cardioïde donnée, celle qui est la plus éloignée du point de rebroussement de cette courbe? *Généralisation* :—Etant donnée la courbe représentée par l'équation $\rho = a\cos^n\omega/n$ en coordonnées polaires, déterminer quelle est, parmi les normales à cette courbe, celle qui est la plus éloignée du pôle?

Solution by C. MORGAN, M.A.; H. J. WOODALL, A.R.C.S.; *and others.*



Let $r = a(1-\cos\theta)$ be the given cardioid; the perpendicular from the focus on the normal

$$= \frac{r}{\{1+r^2(d\theta/dr)^2\}^{\frac{1}{2}}} = 2a\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta.$$

For a maximum value of the perpendicular, $\cos\frac{1}{2}\theta-\cos^3\frac{1}{2}\theta$ must be a maximum; whence

$$\cos\tfrac{1}{2}\theta = \pm\tfrac{1}{3}\sqrt{3}; \quad \text{therefore } r = \tfrac{2}{3}a.$$

General case. $\rho = a\cos^r\omega/n$; whence perpendicular on the normal $= a\cos^n\omega/n\sin\omega/n$. For a maximum

$$1/n\cos^n\omega/n\,.\,\cos\omega/n-\sin^2\omega/n\,.\,\cos^{n-1}\omega/n = 0, \quad \cot^2\omega/n = n,$$

$$r = a\left(\frac{n}{1+n}\right)^{\frac{1}{2}n}.$$

11746. (J. Rice.)—Show that the sum of the series

$$\left\{\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\ldots+1/(2n-1)\right\}+\tfrac{1}{3}\left\{(\tfrac{1}{3})^3+(\tfrac{1}{5})^3+\ldots+\left[1/(2n-1)\right]^3\right\}$$
$$+\ldots+1/(2r-1)\left\{(\tfrac{1}{3})^{2r-1}+(\tfrac{1}{5})^{2r-1}+\ldots+\left[1/(2n-1)\right]^{2r-1}\right\}$$
$$+\ \&\text{c.}\ \ldots\ \text{ad. infin.} = \tfrac{1}{2}\log n.$$

Solution by H. J. Woodall; H. W. Curjel, B.A.; *and others.*

$$\text{Series} = \Sigma_{x=1}^{x=n-1}\left\{\frac{1}{2x+1}+\frac{1}{3(2x+1)^3}+\ldots\right\}$$
$$= \tfrac{1}{2}\Sigma_{x=1}^{x=n-1}\log\frac{x+1}{x} = \frac{\log n}{2}.$$

11788. (Professor Neuberg.)—Trouver

$$\int\frac{dx}{\sin(x+a)\sin(x+b)\sin(x+c)}.$$

Solution by H. W. Curjel, B.A.; A. Kahn, B.A.; *and others.*

$$\int\frac{dx}{\sin(x+a)\sin(x+b)\sin(x+c)} = \frac{4}{\Sigma\sin 2(b-c)}\int\Sigma\frac{\sin(b-c)}{\sin(x+a)}dx$$
$$= \frac{4\Sigma\sin(b-c)\log\tan\tfrac{1}{2}(x+a)}{\Sigma\sin 2(b-c)}+C.$$

9402. (C. M. Goodyear, M.A.)—If A, B, C be the angles of an acute-angled plane triangle, prove that

$$(\tan^2 A)^{\tan^2 A}(\tan^2 B)^{\tan^2 B}(\tan^2 C)^{\tan^2 C} \not< 19683.$$

10615. (H. W. Segar, M.A.)—If, in a triangle, we have $a > b > c$, or $b > c > a$, or $c > a > b$, prove that (1)

$$\left(\frac{\sin B}{\sin C}\right)^{\cos A}\left(\frac{\sin C}{\sin A}\right)^{\cos B}\left(\frac{\sin A}{\sin B}\right)^{\cos C} < 1;$$

and if (2) the triangle be acute-angled, then also

$$\left(\frac{\cot B}{\cot C}\right)^{\sec 2A}\left(\frac{\cot C}{\cot A}\right)^{\sec 2B}\left(\frac{\cot A}{\cot B}\right)^{\sec 2C} < 1.$$

Solution by Professors Zerr, Beyens, *and others.*

(9402.) The expression is a minimum when $A = B = C = 60°$. Then we have $(\tan^2 A)^{\tan^2 A} (\tan^2 B)^{\tan^2 B} (\tan^2 C)^{\tan^2 C} = 19683$, and for any other values the expression > 19683.

(10615.) Both the expressions are a maximum when $A = B = C = 60°$. The expressions then $= 1$. For any other value the expressions then < 1.

11828. (Professor Bénézech.) — On considère la circonférence qui passe par le sommet A et par le point de Lemoine d'un triangle ABC, et qui coupe orthogonalement le cercle circonscrit. Démontrer qu'on a, pour tout point M de cette ligne,

$$a^2 MA^2/(b^2 . MB^2 - c^2 . MC^2) = m_a^2/(m_b^2 - m_c^2),$$

a, b, c désignant les côtés du triangle, m_a, m_b, m_c les médianes.

Solution by Rev. T. R. Terry, M.A.; *and* W. J. Greenstreet, M.A.

The expression equated to zero is the sum of multiples of squares of distances from fixed points; therefore the given equation represents a circle. Since $MA^2 \sin^2 A = \beta^2 + \gamma^2 + 2\beta\gamma \cos A$, and $4m_a^2 = 2b^2 + 2c^2 - a^2$, the equation is

$$\beta^2 + \gamma^2 + 2\beta\gamma \cos A - \lambda (\gamma^2 - \beta^2 + 2\gamma\alpha \cos B - 2\alpha\beta \cos C) = 0,$$

where $$3\lambda (c^2 - b^2) = 2b^2 + 2c^2 - a^2.$$

This is obviously satisfied by $\alpha/a = \beta/b = \gamma/c$.

Therefore the circle passes through the Lemoine point.

Also, the circle passes through A, and the tangent at A is $\beta \cos C - \gamma \cos B = 0$, which passes through the circumcentre.

Therefore the circle is orthogonal to the circumcircle.

9636. (Charles L. Dodgson, M.A.) — If 3 numbers, not in arithmetical progression, be such that their sum is a multiple of 3, prove that the sum of their squares is also the sum of another set of 3 squares, the two sets having no common term.

Solution by Professor G. B. M. Zerr.

Let $3m$, $21m$, $30m$ be the three numbers; then we have

$$3m + 21m + 30m = 3 \times 18m.$$

Also $\qquad (3m)^2+(21m)^2+(30m)^2 = (6m)^2+(15m)^2+(33m)^2$
$= (5m)^2+(13m)^2+(34m)^2 = (10m)^2+(17m)^2+(31m)^2$
$= (14m)^2+(23m)^2+(25m)^2.$

[Mr. DODGSON states that, in this solution, Prof. ZERR "takes a single special instance of 3 numbers, and seems to think that the theorem, since it is true in this single instance, is thereby proved to be true universally." He submits the following theorem, and asks whether Professor ZERR would consider the appended proof a sound logical one.

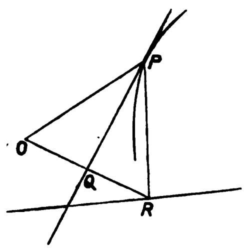
"(*Theorem.*) If 3 numbers be such that their sum is a multiple of 7, the sum of their squares is a multiple of 9.

"(*Proof.*) Let m, $2m$, $11m$ be the 3 numbers. Then $m+2m+11m = 7\times 2m$. Also, $m^2+(2m)^2+(11m)^2 = 126m^2 = 9\times 14m^2$."

We shall be glad to have a further solution of the Question.]

8125. (By Professor SÁRADÁRANJAN RÂY, M.A.)—A parabola has its focus at the centre of a given rectangular hyperbola, and touches the hyperbola; prove that the envelope of its directrix is the Lemniscate of Bernoulli. Generally, if the given curve be $r^m = a^m \cos m\theta$, the envelope is the curve $r^m = (2a)^m \left(\cos \dfrac{m\theta}{m+1}\right)^{m+1}$.

Solution by H. J. WOODALL, A.R.C.S.



Locus of P is the curve, of Q the pedal, of R similar to pedal. R is on the directrix (OQ = QR) of the parabola (focus O), which touches the curve at P. For a consecutive point P_1, we get Q_1 and R_1.

The two directrices cut between R and R_1, and when P_1, in the limit, moves up to P, R_1 moves up to R; hence the envelope of the directrices is locus of R, which is similar to pedal.

Pedal of $r^m = a^m \cos m\theta$ with respect to pole

is $\qquad r^m = a^m \left(\cos \dfrac{m\theta}{m+1}\right)^{m+1}.$

Therefore the required envelope ($r_1 = 2r$) is $r_1{}^m = (2a)^m \left(\cos \dfrac{m\theta}{m+1}\right)^{m+1}.$

Hyperbola $\qquad r^2 \cos 2\theta - a^2 = 0,$

i.e., $\qquad r^{-2} = +a^{-2}\cos 2\theta = a^{-2}\cos\{-2\theta\};$

therefore pedal is

$$r^{-2} = a^{-2}\left[\cos\left\{\frac{-2}{-2+1}\,\theta\right\}\right]^{-1} = a^{-2}\left[\cos 2\theta\right]^{-1}$$
$$= a^{-2}(\cos 2\theta)^{-1},$$

therefore $\qquad r^2 = a^2 \cos 2\theta, \quad$ a lemniscate;

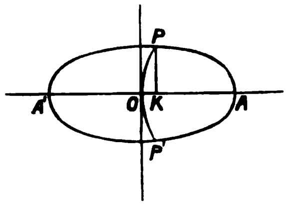
the required locus is $\qquad r^2 = 4a^2\cos 2\theta, \quad$ also a lemniscate.

[It can be easily proved that, whatever the given curve may be, if the

fixed focus of the parabola be the pedal origin, the envelope is always similar to, and similarly placed with, the first pedal of the curve.]

5856. (By Professor MATZ, M.A.)—A point is taken at random within the surface of an ellipse, whose axes are $2a$ and $2b$; find (1) the chance that the distance from the said point to one end of the major axis exceeds a; and (2) the chance that the distance of the said point from the centre of the ellipse exceeds b.

Solution by H. J. WOODALL, A.R.C.S.



(1) Chance $=$ A'POP' $: \pi ab$;

ellipse is $\quad y = \pm b\,(a^2-x^2)^{\frac{1}{2}}/a$;

circle is $\quad y = \pm (2ax-x^2)^{\frac{1}{2}}$.

Ellipse and circle cut at

$$x = \left\{a^3 - a\,(a^4-a^2b^2+b^4)^{\frac{1}{2}}\right\}/(a^2-b^2) = x_1 \text{ say.}$$

Area $\qquad$ APOP' $= \int y\,dx$

$$= 2\int_0^{x_1}(2ax-x^2)^{\frac{1}{2}}\,dx + 2\int_{x_1}^{a}\frac{b\,(a^2-x^2)}{a}\,dx$$

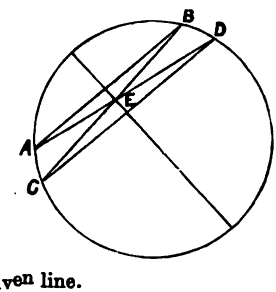
$$= a^2\left[\cos^{-1}\{(a-x)/a\} - (a-x)\{a^2-(a-x)^2\}\;/a^2\right]_0^{x_1} + ab\left[\cos^{-1}(x/a) - x\,(a^2-x^2)^{\frac{1}{2}}/a^2\right]_a^{x_1} = \text{A};$$

$\therefore$ chance $= 1-\text{A}/\pi ab$.

(2) Chance $= 1-b/a\ \left[= (\pi ab-\pi b^2)/\pi ab\right]$.

11537. (MORGAN BRIERLEY.)—AB is a variable chord of a circle, parallel to a line given in position; and parallel to AB another chord CD is drawn; and AD, CB meet in E; find the locus of E.

Solution by Prof. CHAKRIVARTI; R. KNOWLES, B.A.; *and others.*



Let $x^2+y^2=c^2$, $y=mx$ be the equations to the circle and to a line through the centre parallel to the given line. The polar of E is a line through the point of intersection of AC and BD parallel to AB or CD, because it passes through the intersection of AB and CD, and therefore meets them at an infinite distance; the equation to the polar of E $(x'y')$ is $\quad x'x+y'y=c^2$;

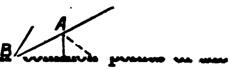
therefore $-x'/y' = m$ or $y = -x/m$

is the required locus, and is a line through the centre at right angles to the given line.

11807. (J. C. St. Clair.)—Given two unequal homographic pencils with different centres, show that (1) if one pencil rotate round its centre, the conics generated in the successive positions by the intersections of corresponding rays have two imaginary points in common; and (2) if both pencils rotate in such a manner as to generate straight lines, these lines envelope a conic.

Solution by Professor Schoute; *the* Proposer; *and others.*

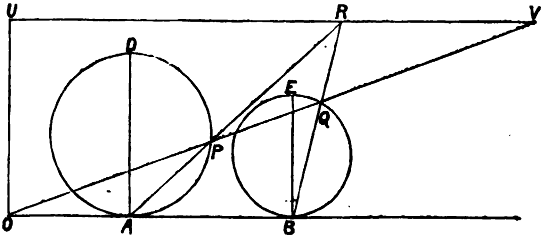
1. Let O, O′ be the given centres, and let the ray OC of the rotating pencil corre-



given conics.

Solution by B. J. Dobbs, B.A.; Professor Farny; *and others.*

Project two of the points in which the given conics intersect into the circules, then the conics become circles, and any conic passing through the common points of the given conics also becomes a circle.



Let AP and BQ meet at R, and draw URV parallel to AB to meet OPQ in V and the perpendicular from O to OA in U. Draw AD, BE diameters at A, B. Then BP/RV = AP/OA and RA/OU = AD/AP; hence

the angle AOB = COP. Then, since AB, PA are equally inclined to OA, O′A (see Fig. 2), B and P are isogonal conjugates with respect to the triangle AOO′, and therefore ∠AO′B = CO′P. Hence A has moved to another point B on AC. Let the intersection of OB, O′B have moved to D. Then, since the segments BA, BD subtend equal angles both at O and O′, and BA touches at A a conic whose foci are O, O′, BD touches the same conic at D. But BD is evidently the line generated by all the corresponding pairs of rays in the new position; and it may also be shown as above to be the locus of intersection of the same pair of rays in each position.

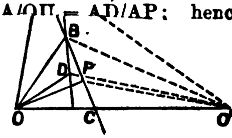


Fig. 2.

The conic will be an ellipse or a hyperbola according as the homography of the pencils is direct, as in Fig. 1, or opposite, as in Fig. 2.

[If a pencil of rays rotates round its centre, its two isotropic rays do not change their position. This proves the first part of the problem. If i_1 and j_1 are the isotropic rays of the rotating pencil, and i_2 and j_2 the

corresponding rays of the other one, the imaginary points (i_1, i_2) and (j_1, j_2) belong to all the generated conics. Therefore these conics form a pencil.

If we pay attention to signs, the correspondence between the angles $\alpha_1 = a_1O_1b_1$ and $\alpha_2 = a_2O_2b_2$ formed by two corresponding couples (a_1, b_1) and (a_2, b_2) of the rotating pencils is a (2, 2) correspondence. This proves the second part of the problem. For, if P is a point chosen at random in the plane of the two rotating pencils, there are two positions for which the couples (O_1P, O_2P), (O_1O_2, O_2O_1) are couples of corresponding rays.]

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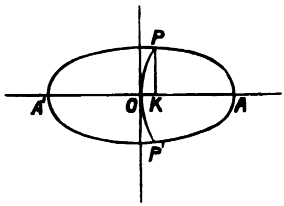
(1) Chance $= A'POP' : \pi ab$;

ellipse is $y = \pm b(a^2-x^2)^{\frac{1}{2}}/a$;

circle is $y = \pm(2ax-x^2)^{\frac{1}{2}}$.



Ellipse and circle cut at

$$x = \left\{a^3 - a(a^4-a^2b^2+b^4)^{\frac{1}{2}}\right\}/(a^2-b^2) = x_1 \text{ say.}$$

Area $\quad APOP' = \int y\,dx$

$$= 2\int_0^{x_1}(2ax-x^2)^{\frac{1}{2}}\,dx + 2\int_{x_1}^{a}\frac{b(a^2-x^2)}{a}\,dx$$

$$= a^2\left[\cos^{-1}\{(a-x)/a\} - (a-x)\{a^2-(a-x)^2\}/a^2\right]_0^{x_1} + ab\left[\cos^{-1}(x/a) - x(a^2-x^2)^{\frac{1}{2}}/a^2\right]_a^{x_1} = A;$$

$\therefore$ chance $= 1 - A/\pi ab$.

(2) Chance $= 1 - b/a\ \left[= (\pi ab - \pi b^2)/\pi ab\right]$.

11537. (MORGAN BRIERLEY.)—AB is a variable chord of a circle, parallel

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11851. (Professor SYLVESTER.) — Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line.

Solution by H. J. WOODALL, A.R.C.S.

Suppose that a set of n points are so situated as to fulfil the condition (but not collinear). Now abstract one of them and note its position; then the $\frac{1}{2}(n-1)(n-2)$ lines joining the remaining points must pass through the position of the abstracted point, which is absurd.

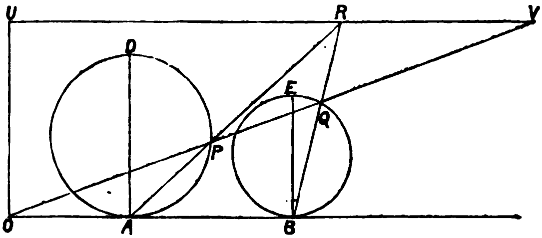
[This does not appear to be a complete proof, *e.g.*, it leaves out of consideration the fact that $\frac{1}{2}(n-1)(n-2)$ is the maximum limit of the number of lines; on the other hand, some of the satisfied lines will be unaltered by abstraction of the point; leaving these out of consideration,

it will still appear absurd. The following method (something after the inductive style) is equally incomplete, but may be worth notice :—Build up the system commencing from the triangle ; now "satisfy" the sides, and thus cause 6 "wants" ; satisfy these at the intersections 2 and 2, causing 6 "wants," and so on.]

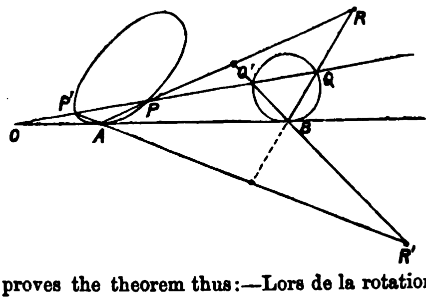
11870. (EDITOR.)—If OAB be a fixed straight line touching two given conics in A, B, OPQ any straight line through O meeting the two conics in P, Q, prove that the locus of the intersection of the straight lines AP, BQ is a conic passing through the four common points of the two given conics.

Solution by B. J. DOBBS, B.A.; Professor FARNY; *and others.*

Project two of the points in which the given conics intersect into the circules, then the conics become circles, and any conic passing through the common points of the given conics also becomes a circle.



Let AP and BQ meet at R, and draw URV parallel to AB to meet OPQ in V and the perpendicular from O to OA in U. Draw AD, BE diameters at A, B. Then $RP/RV = AP/OA$, and $RA/OU = AD/AP$; hence $\dfrac{RP \cdot RA}{OU \cdot RV} = \dfrac{AD}{OA}$; similarly, $\dfrac{RQ \cdot RB}{OU \cdot RV} = \dfrac{EB}{OB}$; thus $\dfrac{RP \cdot RA}{RQ \cdot RB} =$ a constant, and the locus of R is a circle coaxal with the given circles.



[If P′, Q′ be the other points in which the straight line OPQ intersects the given conics, and if AP′, BQ′ intersect in R′; then RR′ passes through a fixed point, namely the pole of AB with respect to the conic, locus of R.

Professor DROZ FARNY proves the theorem thus:—Lors de la rotation

de la transversale, les droites AP et AP′, ainsi que BQ et BQ′, décrivent 2 faisceaux homologiques de centres A et B en involution.

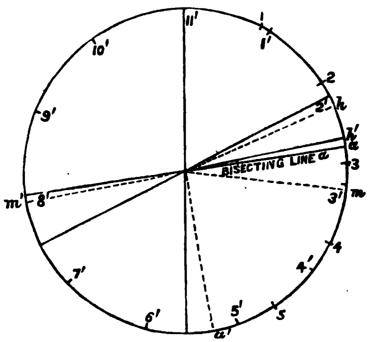
Les rayons correspondants de 2 faisceaux involutifs homologiques se coupent généralement suivant les points d'une quartique admettant les centres A et B comme points doubles.

Dans le problème proposé 2 rayons correspondants coincident avec la ligne des centres. Le lieu se compose donc de la droite AB comptée doublement et d'une conique qui passe évidemment par les 4 points d'intersection des 2 coniques; car, soit C l'un de ces points lorsque la transversale OPQ passe par C, les rayons AC et BC sont correspondants dans les 2 involutions.

On peut aussi, ainsi que l'a fait CHASLES, *Géométrie Supérieure*, p. 482, chercher le lieu lorsque les coniques sont des cercles; on trouve alors un cercle de même axe radical que les premiers; on obtient ensuite le théorème général par projection.]

11837. (D. BIDDLE.)—Find the time indicated by a clock or watch, having given the position (a or $a + 180°$) of that diameter of the dial which bisects the angle separating the hands, and the interval (β) which must elapse before the hands are next in direct opposition. Also show the peculiar interdependence of a and β; either may be *any* part of the hour-circle, but both cannot be.

Solution by MORGAN BRIERLEY; *the* PROPOSER; *and others.*



Let the hour circle of the clock or watch be divided into eleven equal parts of $65\frac{5}{11}$ minutes each; then at every one of these divisions the hour and minute fingers will be exactly together, and when the hour finger is half-way between any two divisions the minute finger will be in opposition to it on the other side of the dial. When the fingers are together, the bisecting diameter a will be in the same line with them; and when they are in opposition it will be, obviously, at right angles to them. The given position of a will determine at which division the fingers were last together. Let, now, a be the given position of the bisecting line, and h the half distance between say 2′ and 3′, the second and third $65\frac{5}{11}$ minutes divisions from the vertex 11′, the end of the twelfth hour division. Suppose h and m to be the *required* positions of the hour and minute fingers, and h' and m' their positions when in opposition; the bisecting line a will then be at a', at right angles to them. In the given interval,

β, a has moved from a, the given position, to a'; consequently aa' is given, and therefore ah'; as also ah, since the velocity of h to a is as 1 to 6, and to m as 1 to 12, and the required time thus found.

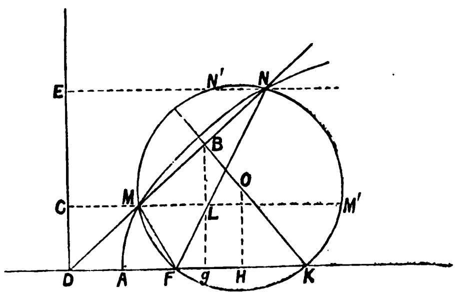
That m and m' can never be simultaneously upon *any* two of the hour divisions is evident from inspection.

[Although the hands meet and are in direct opposition only 11 times in 12 hours, they have their separating angle bisected by a given diameter 13 times in that period, that is to say, every $55\frac{5}{13}$ minutes. Let $x = m-a$; then $a+x = 12(a-x-5h)$, whence $x = \frac{1}{13}(11a-60h)$, and $m = \frac{1}{13}(24a-60h)$, where h can be replaced by 0, 1, 2, ... 12; and when the result is a *minus* quantity, it means so many minutes *to* the specified hour, or $(60-m)$ minutes past the previous one. For example, in the figure, a is at about 14 minutes past the vertical, and it bisects the angle separating the hands at the time required. Therefore $m = \frac{1}{13}(336-60h)$, giving the following: (1) $25\frac{11}{13}$ past 0 or 12; (2) $21\frac{3}{13}$ past 1; (3) $16\frac{8}{13}$ past 2; (4) 12 past 3; (5) $7\frac{5}{13}$ past 4; (6) $2\frac{10}{13}$ past 5; (7) $58\frac{2}{13}$ past 5; (8) $53\frac{7}{13}$ past 6; (9) $48\frac{12}{13}$ past 7; (10) $44\frac{4}{13}$ past 8; (11) $39\frac{9}{13}$ past 9; (12) $35\frac{1}{13}$ past 10; (13) $30\frac{6}{13}$ past 11. To decide which of these is the actual time, β is also given. But, in giving β, our choice is restricted, for $\beta = \frac{12}{11}(30+5h)-m$. Thus, in the case before us, for (1), β must be $6\frac{126}{143}$; for (2) $5\frac{5}{11}+4\frac{8}{13}$ additional, or $16\frac{136}{143}$; for (3) $27\frac{3}{143}$, and so on. But where a and β are correctly given, we have $m = \frac{12}{11}(30+5h)-\beta$ $= \frac{1}{13}(24a-60h)$, whence $h = \frac{1}{1440}(264a+143\beta-4680)$, and m follows. a and β are restricted, one or other, by $143\beta = 4680+1440h-264a$.]

11727. (Professor VUITTENEZ.)—Dans une parabole de foyer F, on mène par le point d'intersection D de l'axe et de la directrice une secante DMN; soient M_1 et N_1 les points de rencontre de la circonférence passant par FMN avec les parallèles à l'axe issues de M et de N. Démontrer que $FM_1 = FN_1$.

11804. (R. KNOWLES, B.A.)—In Quest. 11727, if the circle FMN of centre O meet the axis again in K, prove that (1) OK bisects MN, (2) the locus of O is a semi-cubical parabola.

I. *Solution by* Professor A. DROZ FARNY.



Soient L le point d'intersection de FN avec MM_1, B le milieu de MN, et G, H les projections de B et O sur l'axe. On a

$FL/FN = DM/DN$
$= MC/NE = MF/NF$,

d'où $FL = FM$.

Angle FMM'
$= FLM = FNN'$,

donc
corde $FM' = FN'$.

Mais corde KM = FM′ et corde KN = FN′; donc corde KM = KN, et par conséquent KO est perpendiculaire sur MN en son point milieu.

L'équation de la parabole par rapport à l'axe et à la directrice étant $y^2 = 2px - p^2$ la droite DMN aura pour équation $y = mx$.

Il en résulte pour l'abscisse $DG = p/m^2$. On sait que $GK = FD = p$.

On aura donc, pour les coordonnées du centre O,

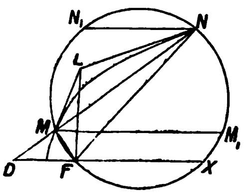
$$FK = DG = p/m^2, \quad FH = p/2m^2, \quad DH = x = p/2m^2 + p,$$
$$OH = y = HK \tan K = FH \,.\, 1/m = p/2m^3.$$

On obtiendra le lieu en éliminant la variable m entre les 2 équations

$$x - p = p/2m^2 \quad \text{et} \quad y = p/2m^3;$$

d'où $py^2 = 2(x-p)^3$, ce qui est bien une parabole semi-cubique.

II. *Solution by* H. W. Curjel, B.A.; Prof. Mukhopadhyay; *and others.*



Let DF meet the circle again in X, and let L be the pole of MN.

Then the angle LFM = the angle LFN, and FL is the polar of D, and is therefore perpendicular to the axis;

$$\therefore\ FMM_1 = DFM = NFX$$
$$= N_1NF;$$

$\therefore$ the chord FM_1 = the chord FN_1.

5518. (Rev. W. Roberts, M.A.)—Let S denote the length of the periphery of an ellipse; S_1, S_2 the length of its first two positive pedals, and S_{-1}, S_{-2} the lengths of its first two negative pedals; then, if the origin be at the centre of the ellipse, prove that

$$(S_1 + S_{-1})\, S_{-1} = (2S - S_2)(3S - S_{-2}).$$

Solution by Professor Sebastian Sircom, M.A.

We have $S = \int_0^{2\pi} p\,d\omega$. In the ellipse, $r^2 = a^2 + b^2 - \dfrac{a^2b^2}{p^2}$; also

$$r^2 = p^2 + (dp/d\omega)^2;$$

hence we have

$$S = 4\int_b^a \frac{p\,dp}{(r^2-p^2)^{\frac12}} = 4\int_b^a \frac{r\,dr}{(r^2-p^2)^{\frac12}}; \qquad S_1 = \int_0^{2\pi} r\,d\omega = 4\int_b^a \frac{r\,dp}{(r^2-p^2)^{\frac12}}.$$

Let p_1, r_1 be the perpendicular from the origin and radius vector of S_1; then $p_1 = p^2/r$, $r_1 = p$, and

$$S_2 = 4\int_b^a \frac{r_1\,dp_1}{(r_1^2-p_1^2)^{\frac12}} = 4\int_b^a \frac{2p\,dp}{(r^2-p^2)^{\frac12}} - 4\int_b^a \frac{p^2}{r}\cdot\frac{dr}{(r^2-p^2)^{\frac12}};$$

whence $2S - S_2 = 4\displaystyle\int_b^a \frac{p^2}{r}\cdot\frac{dr}{(r^2-p^2)^{\frac12}}$. Similarly, $S_{-1} = 4\displaystyle\int_b^a \frac{p\,dr}{(r^2-p^2)^{\frac12}}$,

and $$3S - S_{-2} = 4\int_b^a \frac{p\,dp}{(r^2-p^2)^{\frac{1}{2}}} + 4\int_b^a \frac{r^2}{p}\cdot\frac{dp}{(r^2-p^2)^{\frac{1}{2}}}.$$

$$\int_b^a \frac{r^2}{p}\cdot\frac{dp}{(r^2-p^2)^{\frac{1}{2}}} = \int_b^a \left(a^2+b^2-\frac{a^2b^2}{p^2}\right)\frac{dp}{p\,(r^2-p^2)^{\frac{1}{2}}}$$
$$= (a^2+b^2)\int_b^a \frac{dp}{p\,(r^2-p^2)^{\frac{1}{2}}} - \int_b^a \frac{r\,dr}{(r^2-p^2)^{\frac{1}{2}}};$$

whence $$3S - S_{-2} = 4\,(a^2+b^2)\int_b^a \frac{dp}{\left[(a^2-p^2)\,(p^2-b^2)\right]^{\frac{1}{2}}};$$

$$2S - S_2 = 4a^2b^2\int_b^a \frac{dr}{r\,(a^2+b^2-r^2)^{\frac{1}{2}}\,(a^2-r^2)^{\frac{1}{2}}\,(r^2-b^2)^{\frac{1}{2}}}.$$

Putting $p = ab/r_1$,

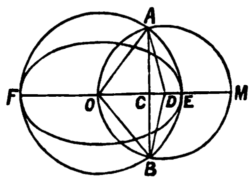
$$S_1 + S_{-1} = 4\int_b^a \frac{d\,(pr)}{(p^2-r^2)^{\frac{1}{2}}} = 4\,(a^2+b^2)\int_b^a \frac{dp}{r\,(r^2-p^2)^{\frac{1}{2}}}$$
$$= 4ab\,(a^2+b^2)\int_b^a \frac{dr_1}{r_1\,(a^2+b^2-r_1^2)^{\frac{1}{2}}\,(a^2-r_1^2)^{\frac{1}{2}}\,(r_1^2-b^2)^{\frac{1}{2}}};$$

$$S_{-1} = 4\int_b^a \frac{p\,dr}{(r^2-p^2)^{\frac{1}{2}}} = 4ab\int_b^a \frac{dr}{\left[(a^2-r^2)\,(r^2-b^2)\right]^{\frac{1}{2}}};$$

and, since the limits are everywhere the same, we obtain the required result by multiplication.

7207. (Professor ORCHARD, M.A.)—A fixed circle passes through the centre of the ellipse $r = l/(1+e\cos\theta)$, and has the same area. The ellipse revolves round an axis through its centre perpendicular to its plane. Find, for a single revolution, the area common to the two curves.

Solution by Professor G. B. M. ZERR.



In a revolution the ellipse describes the circle AFBE. Although the whole area of the ellipse will have passed over the area AFBE, yet the area common to both is the area AOBE.

Area of ellipse $= \pi l^2/(1-e^2)^{\frac{3}{2}}$;

$\therefore$ DA $= r = l/(1-e^2)^{\frac{3}{4}}$, AO $= \mathrm{R} = l/(1-e^2)$,

$$\mathrm{AC} = c = \frac{\mathrm{R}\,(4r^2-\mathrm{R}^2)^{\frac{1}{2}}}{2r}.$$

Area AOEB $= \mathrm{R}^2\left\{\sin^{-1}c/\mathrm{R} - c/\mathrm{R}^2.(\mathrm{R}^2-c^2)^{\frac{1}{2}}\right\} + r^2\left\{\sin^{-1}c/r - c/r^2.(r^2-c^2)^{\frac{1}{2}}\right\}$,

$$\frac{c}{\mathrm{R}} = \frac{\sqrt{4\,(1-e^2)^{\frac{1}{2}}-1}}{2(1-e^2)^{\frac{1}{4}}},\quad \frac{c}{r} = \frac{\sqrt{4\,(1-e^2)^{\frac{1}{2}}-1}}{2\,(1-e^2)^{\frac{1}{2}}},\quad \frac{c}{\mathrm{R}^2}\,(\mathrm{R}^2-c^2)^{\frac{1}{2}} = \frac{\sqrt{4\,(1-e^2)^{\frac{1}{2}}-1}}{4\,(1-e^2)^{\frac{1}{2}}},$$

$$\frac{c}{r^2}\,(r^2-c^2)^{\frac{1}{2}} = \frac{\left\{2\,(1-e^2)^{\frac{1}{2}}-1\right\}\sqrt{4\,(1-e^2)^{\frac{1}{2}}-1}}{4\,(1-e^2)};$$

$$\therefore \text{ area AOEB} = \frac{l^2}{(1-e^2)^2}\sin^{-1}\frac{\sqrt{4(1-e^2)^{\frac12}-1}}{2(1-e^2)^{\frac14}}$$

$$+\frac{l^2}{(1-e^2)^{\frac32}}\sin^{-1}\frac{\sqrt{4(1-e^2)^{\frac12}-1}}{2(1-e^2)^{\frac14}}-\frac{l^2}{2(1-e^2)}\sqrt{4(1-e^2)^{\frac12}-1}$$

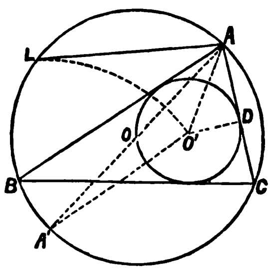
$$= a^2\sin^{-1}\tfrac12\left\{(4b-a)/b\right\}^{\frac12}+ab\sin^{-1}1/2b\,(4ab-a^2)^{\frac12}-\tfrac12 b^2\left\{(4b-a)/a\right\}^{\frac12},$$

where a and b are the semi-axes of the ellipse;

$$\therefore \text{ area AOEB} = a(a+2b)\sin^{-1}\tfrac12\left\{(4b-a)/b\right\}^{\frac12}-\tfrac12 b^2\left\{(4b-a)/a\right\}^{\frac12}.$$

11871. (R. Tucker, M.A.) (O), (O′), are the circum- and in-circles of the triangle ABC, and A′B′C′ is the diametraltriangle; prove that (1) the sum of the squares of the tangents (taken once) from the six vertices to (O′) = $6(2R^2-2Rr-r^2)$; (2) the circle, centre A′, radius A′O′, cuts (O) in L; (3) AL is a mean proportional between AB, AC.

Solution by Professor A. Droz Farny, W. J. Dobbs, M.A., *and others.*



1. Soient t et t' es tangentes de A et $A'a^2$ (O′).

$$t^2 = (AO_1)^2 - r^2, \quad t_1^2 = (A_1O_1)^2 - r$$

$$t^2 + t_1^2 = (AO_1)^2 + (A_1O_1)^2 - 2r^2$$

$$= 2R^2 - 2r^2 + 2(OO_1)^2$$

$$= 4R^2 - 4Rr - 2r^2,$$

$$\Sigma t^2 + t_1^2 = 6(2R^2 - 2Rr - r^2).$$

2, 3. $(AL)^2 = (AA_1)^2 - (A_1O_1)^2$

$$= 4R^2 - (A_1O_1)^2;$$

mais $(AO_1)^2 + (A_1O_1)^2 = 4R^2 - 4Rr,$

et $$(AO_1)^2 = \frac{(AD)^2}{\cos^2\frac12 A} = \frac{(p-a)^2}{\cos^2\frac12 A} = \frac{bc(p-a)}{p}.$$

Donc $(AL)^2 = 4Rr + bc(p-a)/p.$

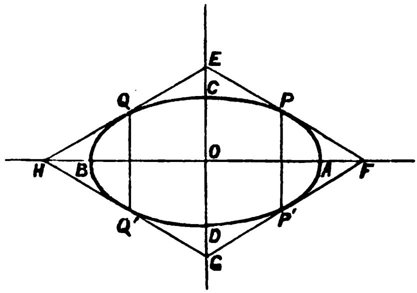
Mais $4Pr = abc/$ $(AL)^2 = abc/p + bc(p-a)/p$ $(AL)^2 = bc.$

11064. (W. J. Greenstreet, M.A.)—A parallelogram is formed by joining the vertices of an ellipse. Find (1) the points of contact of an ellipse inscribed to this parallelogram and confocal with the original ellipse; (2) the radii of curvature at these points; and (3) the area of circles osculating at these points.

Solution by Professors ZERR, SHIELDS, *and others.*

Let $\left.\begin{matrix} OH = OF = b \\ OE = OG = a \end{matrix}\right\}$ be the semi-axes of the given ellipse,

$\left.\begin{matrix} OB = OA = \lambda \\ OC = OD = \beta \end{matrix}\right\}$ be the semi-axes of the confocal ellipse.



Then the area of

$FGHE = 4\lambda\beta = 2ab$;

$\therefore\ 2\lambda\beta = ab$,

and $\lambda^2 - \beta^2 = a^2 - b^2$.

From these equations we get

$\beta^2 = \frac{1}{2}\left\{(a^4 - a^2b^2 + b^4)^{\frac{1}{2}} - (a^2 - b^2)\right\}$, $\lambda^2 = \frac{1}{2}\left\{(a^4 - a^2b^2 + b^4)^{\frac{1}{2}} + (a^2 - b^2)\right\}$.

Let (x', y') be the coordinates of point P. Then the equation to EPF is $\lambda^2 yy' + \beta xx' = \lambda^2\beta^2$.

Making $y = 0$, we get $x = OF = a = \lambda^2/x'$, or $x' = \lambda^2/a$.

Similarly, $OE = b = \beta^2/y'$, or $y' = \beta^2/b$; therefore coordinates of P are $(\lambda^2/a,\ \beta^2/b)$; of P', $(\lambda^2/a,\ -\beta^2/b)$; of Q, $(-\lambda^2/a,\ \beta^2/b)$; of Q', $(-\lambda^2/a,\ -\beta^2/b)$.

2. The radius of curvature at any point on the ellipse is, from the r and p equation, $(\lambda^2\beta^2)/p^3$.

$$\text{But } p = \frac{\lambda^2\beta^2}{(\lambda^4y'^2 + \beta^4x'^2)^{\frac{1}{2}}} = \frac{\lambda^2\beta^2}{\left\{(\lambda^4\beta^4)/b^2 + (\lambda^4\beta^4)/a^2\right\}^{\frac{1}{2}}}; \quad \therefore\ p = \frac{ab}{(a^2+b^2)^{\frac{1}{2}}},$$

$$\text{and } \rho = \text{radius of curvature} = \frac{\lambda^2\beta^2(a^2+b^2)^{\frac{3}{2}}}{a^3b^3}; \quad \therefore\ \rho = \frac{(a^2+b^2)^{\frac{3}{2}}}{4ab}.$$

$$\text{3. Area of osculating circle} = \pi\rho^2 = \frac{\pi(a^2+b^2)^3}{16a^2b^2}.$$

10660. (Professor SCHOUTE.)—Given four complanar conics: show that there are to be found three right lines that meet these four conics in four couples of points belonging to the same quadratic involution.

Solution by Professor A. DROZ FARNY.

On sait qu'une conique fixe U est coupée par les diverses coniques d'un faisceau MNPQ suivant des cordes qui enveloppent une courbe de troisième classe qui est tangente aux côtés et aux diagonales du quadrilatère inscrit MNPQ, et qu'une droite quelconque est toujours coupée par un faisceau de coniques suivant des couples de points en involution.

Soient A, B, C et D les quatre coniques données; A et B déterminent un faisceau de base MNPQ dont les diverses coniques coupent la conique fixe C suivant des cordes enveloppant une courbe de troisième classe C_3.

Chacune de ces cordes contient 3 couples de points en involution, ses points de rencontre avec les coniques A, B, C.

De même les coniques du faisceau A, B coupent la conique fixe D suivant des cordes enveloppant une seconde courbe de troisième classe C'_3 et contenant chacune 3 couples de points en involution, ses points d'intersection avec les coniques A, B, et D.

Une involution étant déterminée par 2 couples de points, les tangentes communes aux 2 courbes C_3 et C'_3 couperont les 4 coniques proposées suivant 4 couples de points en involution.

Or 2 courbes de troisième classe admettent 9 tangentes communes dont il faut défalquer dans le problème proposé les 4 côtés et les 2 diagonales du quadrilatère inscrit MNPQ qui ne répondent pas à la question.

Il existe donc bien 3 droites qui coupent 4 coniques données suivant 4 couples de points en involution.

[Prof. Schoute remarks that this solution may be put in the following compacter form :—Soient A^2, B^2, C^2, D^2 les quatres coniques données. Les droites qui coupent, A^2, B^2, C^2 en trois couples de points d'une même involution, envellopent une courbe C^3_δ de la troisième classe, dont les coniques dégénérées du réseau (A^2, B^2, C^2) forment un système de tangentes conjuguées. De même les droites qui jouent le même rôle par rapport à A^2, B^2, D^2, enveloppent une C^3_γ. Ces deux courbes C^3_γ et C^3_δ sont touchées par les trois couples de droites du faisceau (A^2, B^2). Donc elles démontrent par ses trois autres tangentes communes, qu'il y a trois droites qui coupent les quatre coniques données en quatre couples de points d'une même involution quadratique.]

11579. (R. Knowles, B.A.)—From the vertices of the triangle ABC, three concurrent lines are drawn to meet the opposite sides in D, E, F, respectively. Prove that the three points of intersection of BC, AC, AB with FE, FD, DE respectively are collinear.

Solution by T. W. K. Clarke ; J. Burke, B.A. ; *and others.*

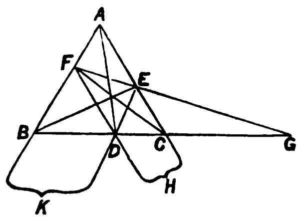
Let EF, BC cut in G, and DF, AC in H ; DE, AB in K ; then we have

$$\frac{AE}{EC}\cdot\frac{BF}{FA}\cdot\frac{CG}{GB} = -1,$$

and two similar equations. But

$$\frac{AE}{EC}\cdot\frac{BF}{FA}\cdot\frac{CD}{DB} = 1 ;$$

hence, multiplying the first set of equations and the two similar ones together,

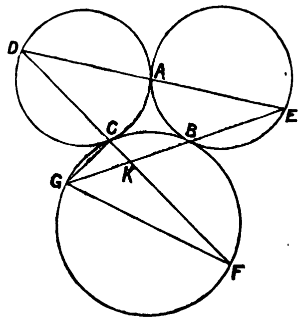


$$\frac{CG}{GB}\cdot\frac{AH}{HC}\cdot\frac{BK}{KA} = -1 ;$$ thus GHK are collinear.

[*Otherwise* :—Since the lines that join pairs of vertices of ABC, DEF are concurrent, the triangles are in perspective hence the intersections of corresponding sides are in a straight line.]

11840. (A. Kahn, B.A.) — Three circles touch one another, A, B, and C being the points of contact. Any line DAE is drawn, cutting again the two circles which touch at A, in D and E respectively. EB and DC are drawn to meet the third circle again in G and F respectively. Prove that GF is a diameter of the third circle.

Solution by W. J. Dobbs, B.A.; W. J. Constable, M.A.; *and others.*



Since AC, AB, BC subtend, at the circumferences of their respective circles, angles which together make up one right angle,

i.e., $\angle$ CDA + BEA + CGB = right angle,

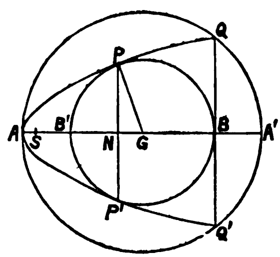
$\therefore$ $\angle$ CKG + CGK = right angle;

$\therefore$ GCK is a right angle

$\therefore$ GF is a diameter.

11712. (Morgan Brierley.)—Prove, geometrically, that the sum of the double ordinate and abscissa of a parabola is equal to the sum of the diameters of the inscribed and circumscribed circles.

Solution by H. W. Curjel, B.A.



Let QPAP′Q′ be the parabola, ANBA′ being the axis, and QBQ′ the double ordinate, and S the focus. Let AQA′Q′ be the circumcircle, and BPB′P′ the incircle cutting the axis in B and B′, and let the normal at P cut the axis in G.

Then $AB \, . \, BA' = QB^2 = AB \times 4AS$;

$\therefore \; BA' = 4AS.$

Also, $NG = 2AS$,

$GB^2 = PG^2 = NG^2 + PN^2$

$= 4AS \, . \, AN + 4AS^2$,

and $BN^2 = GB^2 + 4AS \, . \, GB + 4AS^2$

$= 4AS \, . \, AN + 8AS^2 + 4AS \, . \, GB = 4AS \, . \, AN + 4AS \; NG + 4AS \, . \, GB$

$= 4AS \, . \, AB = QB^2$;

$\therefore \; BN = QB; \qquad \therefore \; QQ' = BB' + 4AS = BB' + BA'$;

$\therefore \; QQ' + AB = BB' + AB + BA' = BB' + AA.$

11824. (Professor GAY.)—On donne deux ellipses dont l'une est intérieure à l'autre. Prouver que les conditions nécessaires et suffisantes pour qu'une sécante quelconque les coupe suivant deux cordes CF, DE, de telle façon qu'on ait toujours CD = EF.

Solution by W. J. DOBBS, B.A.; Professor ZERR; *and others.*

Projecting one ellipse orthogonally into a circle, it is evident that the other must project into a concentric circle. Therefore the two ellipses must be concentric, similar, and similarly situated.

11829. (Professor MANDART.)—Démontrer l'identité

$$(a\cos C + z\sin A)(b\cos A + x\sin B)(c\cos B + y\sin C)$$
$$= (a\cos B + y\sin A)(b\cos C + z\sin B)(c\cos A + x\sin C),$$

a, b, c, A, B, C étant les côtés et les angles d'un triangle.

Solution by Rev. T. R. TERRY, M.A.; W. DICKSON; *and others.*

Since $\dfrac{a\cos C + z\sin A}{b\cos C + z\sin B} = \dfrac{a}{b}$, $\therefore$ $\dfrac{\text{Left hand exp.}}{\text{Right hand exp.}} = \dfrac{a}{b}\cdot\dfrac{b}{c}\cdot\dfrac{c}{a} = 1.$

11868. (Rev. Dr. KOLBE.)—Find a short method of reducing to decimals fractions whose denominator ends in 9; *e.g.*, $\frac{3}{19}$, $\frac{7}{39}$, &c.

Solution by R. CHARTRES; W. J. GREENSTREET, M.A.; *and others.*

Let r = radix of any scale; then $1/(nr-1)$ can be reduced to a recurring fraction by putting 1 for the last figure of the period, and then multiplying from the end by n until the period recurs, thus

$\frac{1}{19} = \cdot\dot{0}5$, &c., $6842\dot{1}$, putting down 1, and multiplying by 2;

$\therefore$ $\frac{3}{19} = \cdot\dot{1}5$, &c., $26\dot{3}$, ,, 3, ,, ,, 2;

$\frac{1}{39} = \cdot\dot{0}2564\dot{1}$, ,, 1, ,, ,, 4;

$\frac{7}{39} = \cdot\dot{1}7948\dot{7}$, ,, 7, ,, ,, 4.

The proof is obvious. This Mr. CHARTRES gave in *Nature* in 1878.

[The PROPOSER remarks that the process is a sort of skew division, the divisor being the number of tens on which the denominator is verging: for 19 divide by 2, for 39 by 4, &c. The division is performed by constructing our dividend as we go along, putting each remainder (not under the next figure, as in ordinary division, but) under the figure we have just put down. Thus, for $\frac{3}{19}$, since 2 into 3 gives 1 and 1 over, the next

thing we divide into is 11, and so on, as thus,

$$\begin{array}{l} 2\,\underline{)\,3\cdot\qquad\qquad} \\ \quad\cdot 157894736\ \&c. \\ \quad\ 1111\ 1\ 11 \end{array}$$

The whole trick consists in looking *upward* instead of *forward* for the next bit of dividend. It reads:—2 into 11, 5 and 1 over; 2 into 15, 7 and 1 over; &c.

The process may often be further shortened by the principle that whenever one stage is a multiple or a measure of a preceding one, we may finish our work by multiplying or dividing in that proportion. Take, for instance, $\frac{7}{89}$: begin with skew division by 9.

$$\begin{array}{l} \frac{7}{89} = \cdot 07865168539325842696 62\ \&c. \\ \qquad 775416743 \end{array}$$

At the 9th figure we have $\binom{5}{3}$ or 35, which is the half of $\binom{0}{7}$ or 70: hence the rest of the decimal is got by dividing all that precedes by 2. It circulates in 44 figures, and of course the second half of the repetend can be written down without any division at all.

If the denominator ends in 99, a skew division by the number of *hundreds*, putting the remainder *one step back*, will give us an easy result; *e.g.*, $\frac{277}{399}$ would be done as follows:—

$$\begin{array}{l} 4\,\underline{)\,277\cdot\qquad\qquad\qquad\qquad} \\ \quad\cdot\dot{6}9423558897243107\dot{7}69 \\ \quad\ 3\ 1\ 1223332\ 11\ \ 33231 \end{array}$$

which reads: 4 into 27, 6 after the decimal point and 3 over; put the 3 one place back, and then 4 into 37, 9 and 1 over; into 16, 4; into 9, 2 and 1 over, put back; &c. Obviously the trick can be extended.]

11880. (A. J. Pressland, M.A.)—If from a point P three normals PQ, PR, PS be drawn to a parabola QRS, and the orthocentre of the triangle formed by the tangents at Q, R, S be O, prove that PO is perpendicular to the directrix.

Solution by R. Tucker, M.A.; W. J. Dobbs, M.A.; *and others.*

If the coordinates of a point on the parabola $y^2-4ax=0$ are $(am^2, 2am)$, then the ordinates of the orthocentre of the tangent triangle and of P are (a_1m_1, m_2m_3); hence, &c. (see Mr. Tucker's "Some Properties of Co-normal Points on a Parabola," *Proc. of Lond. Math. Soc.*, Vol. xxi., pp. 442–451, §§ 13, 15).

10670. (J. Griffiths, M.A.)—Prove that, if $x = \xi + \lambda\eta$, $y = \eta$,

$$A_n = a_n + na_{n-1}\lambda + \frac{n \,.\, n-1}{1\,.\,2} a_{n-2}\lambda^2 + \ldots,$$

where $a_n, a_{n-1}, \ldots$ are functions of x, y, $A_n, A_{n-1}, \ldots$, the corresponding functions of ξ, η, such that

$$\frac{da_n}{dx} = a_0 a_{n+1} - a_1 a_n, \quad \frac{da_n}{dy} = \tfrac{1}{2}(a_0 a_{n+2} - a_2 a_n),$$

$$\frac{dA_n}{d\xi} = A_0 A_{n+1} - A_1 A_n, \quad \frac{dA_n}{d\eta} = \tfrac{1}{2}(A_0 A_{n+2} - A_2 A_n),$$

then
$$\frac{dA_n}{d\eta} = \tfrac{1}{2}(A_0 A_{n+2} - A_2 A_n)$$

$$= \left(\frac{d}{dy} + \lambda \frac{d}{dx}\right)\left(a_n + n a_{n-1}\lambda + \frac{n \cdot n-1}{1 \cdot 2} a_{n-2}\lambda^2 + \ldots\right).$$

Solution by Professor ZERR.

From
$$\frac{da_n}{dx} = a_0 a_{n+1} - a_1 a_n, \quad \frac{da_n}{dy} = \tfrac{1}{2}(a_0 a_{n+2} - a_2 a_n),$$

we have
$$\frac{d}{dx} = a_1(a_0 - 1), \quad \lambda\frac{d}{dx} = \lambda a_1 (a_0 - 1), \quad \frac{d}{dy} = \tfrac{1}{2} a_2 (a_0 - 1);$$

therefore
$$\frac{d}{dy} + \lambda \frac{d}{dx} = \tfrac{1}{2}(a_2 + 2a_1\lambda)(a_0 - 1) \quad\ldots\ldots\ldots(1);$$

also
$$\frac{dA_n}{d\eta} = \tfrac{1}{2} A_2 (A_0 - 1) A_n \quad\ldots\ldots\ldots(2).$$

From the equation
$$A_n = a_n + n a_{n-1}\lambda + \frac{n \cdot n-1}{1 \cdot 2} a_{n-2}\lambda^2 + \ldots \quad\ldots\ldots\ldots(3),$$

we get
$$A_0 = a_0, \quad A_2 = a_2 + 2a_1\lambda;$$

therefore
$$\frac{dA_n}{d\eta} = \tfrac{1}{2}(a_2 + 2a_1\lambda)(a_0 - 1) A_n = \left(\frac{d}{dy} + \lambda\frac{d}{dx}\right) A_n, \text{ from (1)},$$

$$= \left(\frac{d}{dy} + \lambda \frac{d}{dx}\right)\left(a_n + n a_{n-1}\lambda + \frac{n \cdot n-1}{1 \cdot 2} a_{n-2}\lambda^2 + \ldots\right), \text{ from (3)}.$$

11761. (The late Professor WOLSTENHOLME, Sc.D.)—In a given parabola $y^2 = 4ax$, PQ is a chord normal at P, and QX is the perpendicular from Q on the directrix; a curve is traced out by a point whose coordinates are equal to XQ, QP respectively; prove that this curve will be the tricusp quartic $u^{-\frac{1}{2}} + v^{-\frac{1}{2}} + w^{-\frac{1}{2}} = 0$, where

$$u \equiv 4x + 2\sqrt{3}\,y, \quad v \equiv 4x - 2\sqrt{3}\,y, \quad w \equiv x - 9a.$$

Also the equation is $2y^2 = x^2 + 18ax - 27a^2 \pm \{(x-a)(x-9a)^3\}^{\frac{1}{2}}$; the cusps are the points $(0, 0)$, $(9a, \pm 6\sqrt{3a})$; the rectilinear asymptotes $y = \pm(x+a)$; the parabolic asymptote $y^2 = 16a(x-2a)$. The curve cuts the parabolic asymptote when $x = 10a$, $y^2 = 128a^2$; and the rectilinear asymptote when $x^2 - 14ax - 3a^2 = 0$.

Solution by H. W. CURJEL, B.A.; W. J. GREENSTREET, M.A.; *and others*.

Let the coordinates of P be $\frac{a}{m^2}$, $\frac{2a}{m}$; normal is

$$m\left(y-\frac{2a}{m}\right)+x-\frac{a}{m^2}=0;$$

therefore the coordinates of Q are $a\left(\frac{2m^2+1}{m}\right)^2$, $\frac{-2a\,(2m^2+1)}{m}$.

If x, y are coordinates of the corresponding points on the curve,

$$y=\pm\frac{4a\,(m^2+1)^{\frac{3}{2}}}{m},\quad x=\frac{(4m^2+1)(m^2+1)}{m^2}\,a;$$

therefore $$u=4x+2\sqrt{3}\,y=\frac{4a\,(m^2+1)}{m^2}\left\{(m^2+1)^{\frac{1}{2}}+\sqrt{3}\,m\right\}^2,$$

$$v=4x-2\sqrt{3}\,y=\frac{4a\,(m^2+1)}{m^2}\left\{(m^2+1)^{\frac{1}{2}}-\sqrt{3}\,m\right\}^2,$$

$$w=x-9a=\frac{a\,(2m^2-1)^2}{m^2}.$$

Hence we see that the equation may be written in the form

$$u^{-\frac{1}{2}}+v^{-\frac{1}{2}}+w^{-\frac{1}{2}}=0.$$

This represents a tricusp quartic having its cusps at the angles of the triangle formed by $u=0$, $v=0$, $w=0$; *i.e.* at $(0,0)$, $(9a, \pm 6\sqrt{3}\,a)$. Writing the equation

$$y^2\,(y+x)(y-x)-15ay^2x+16ax^3+27a^2y^2=0,$$

we see that the approximations at infinity are

$$y-x-a=0,\quad y+x+a=0,\quad y^2-16a\,(x-2a)=0.$$

The equation to the curve may be written

$$\left\{y^2-16a\,(x-2a)\right\}\left\{y^2-(x+a)^2\right\}-4a^2y^2+48a^3x+3a^4=0;$$

therefore $y^2-16a\,(x-2a)=0$ meets the curve where $x=10a, y^2=128a^2$; and $y^2-(x+a)^2=0$ meets the curve where $x^2-10ax-7a^2=0$.

3919. (Professor HUDSON, M.A.)—A man's expenses exceed his income by £a per annum; he borrows at the end of every year enough to meet this, and, after the first year, to pay the interest on his previous borrowings, the rate of interest at which he borrows increasing each year in geometrical progression, whose common ratio is λ, till, at the end of the n years, it is cent. per cent. What does he then borrow?

Solution by H. J. WOODALL, A.R.C.S.

He borrowed at the end of the first, second, third, nth years respectively the sums £a, £$(a+ra)$ = £$a\,(1+r)$,

$$£\left\{a+ar+a\,(1+r)\,\lambda r\right\}=£a\,(1+r)(1+\lambda r),$$

and £$a\,(1+r)(1+\lambda r)\ldots(1+\lambda^{n-2}r)$.

Interest on this latter is cent. per cent., *i.e.*, $\lambda^{n-1}\,r=1$.

11839. (W. J. JOHNSTON, M.A.)—Prove the following relation between six points A, B, C, D; I, J on a conic. If

$$(12) \equiv (\text{area AIB . area AJB})^{\frac{1}{2}}, \text{ \&c.,}$$

then $$(12).(34)+(23).(14)+(31).(24) = 0.$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

From the anharmonic theory of conics, if AIB represents the area of the triangle AIB, we have

$$\frac{\text{AIB . CID}}{\text{AJB . CJD}} = \frac{\text{AIC . BID}}{\text{AJC . BJD}} = \frac{\text{AID . BIC}}{\text{AJD . BJC}} = r^2,$$

whence $(12)(34) = (\text{AIB . AJB . CID . CJD})^{\frac{1}{2}} = \text{AJB . CJD}r.$

Similarly, $(23)(14) = \text{AJD . BJC}r$; $(31)(24) = \text{CJA . BJD}r$.

But A, B, C, D, J are five points in a plane, therefore

$$\text{AJB . CJD} + \text{AJD . BJC} + \text{CJA . BJD} = 0;$$

whence the result at once follows.

11689. (Professor MORLEY.)—Prove $\sum_1^\infty \frac{1}{2^n \cdot n^2} = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2.$

Solution by H. W. CURJEL, B.A.; Professor ZERR; *and others.*

Let $\Sigma_1^\infty (2^n . n^2)^{-1} = S$; then we have

$$2S - \frac{\pi^2}{6} = \Sigma_1^\infty \left(\frac{2}{2^n . n^2} - \frac{1}{n^2}\right) = -2\int_0^{\frac{1}{2}} \frac{\log(1-x)}{x}dx + \int_0^1 \frac{\log(1-x)}{x}dx$$

$$= -\int_0^{\frac{1}{2}} \frac{\log(1-x)}{x}dx + \int_{\frac{1}{2}}^1 \frac{\log(1-x)}{x}dx$$

$$= -\Big[\log(1-x)\log x\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{\log x}{1-x}dx + \int_{\frac{1}{2}}^1 \frac{\log(1-x)}{x}dx$$

$$= -\Big[\log(1-x)\log x\Big]_0^{\frac{1}{2}} - \int_{\frac{1}{2}}^1 \frac{\log(1-y)}{y}dy + \int_{\frac{1}{2}}^1 \frac{\log(1-x)}{x}dx$$

(where $y = 1-x$)

$$= -\Big[\log(1-x)\log x\Big]_0^{\frac{1}{2}}$$

$= -(\log\frac{1}{2})^2$, for $x^r \log x = 0$ when $x = 0$ for any value of r from 1 to ∞,

$= -(\log 2)^2$;

hence we have, finally, $$S = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2.$$

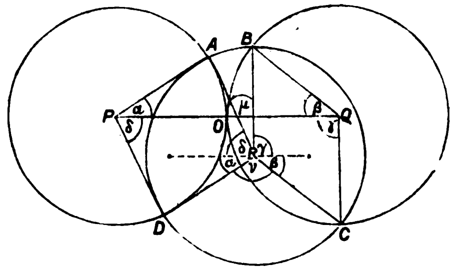
11821. (Professor CROFTON, F.R.S.)—Two equal circles AOD, BOC are cut by a third equal circle ABCD, the two former touching each other at O, a point internal to the third (radius = 1). If we put α, β, γ, δ, μ, ν, for the arcs OA, OB, OC, OD, AB, CD, prove that

$$\mu+\gamma+\delta = \nu+\alpha+\beta = 180^\circ, \quad \cos\mu+\cos\gamma+\cos\delta = \cos\nu+\cos\alpha+\cos\beta,$$

$$\cos\alpha+\cos\beta+\cos\gamma+\cos\delta = 2.$$

Solution by W. J. DOBBS, B.A.; H. W. CURJEL, B.A.; *and others.*

If P, Q, R be centres of the circles, each of the figures ARDP, RBQC is a rhombus; hence



$$\mu+\gamma+\delta = \nu+\alpha+\beta = 180^\circ.$$

Projecting the lines PARBQ in the direction PQ, and also perpendicular to PQ, we have

$$\cos\alpha+\cos\beta+\cos\gamma+\cos\delta = 2, \quad \sin\alpha+\sin\gamma = \sin\beta+\sin\delta.$$

These four equations contain all the relations between α, β, γ, δ, μ, ν. The last two equations may be written thus—

$$\cos\tfrac{1}{2}(\alpha+\gamma)\cos\tfrac{1}{2}(\alpha-\gamma)+\cos\tfrac{1}{2}(\beta+\delta)\cos\tfrac{1}{2}(\beta-\delta) = 1,$$

$$\sin\tfrac{1}{2}(\alpha+\gamma)\cos\tfrac{1}{2}(\alpha-\gamma) = \sin\tfrac{1}{2}(\beta+\delta)\cos\tfrac{1}{2}(\beta-\delta);$$

$$\therefore\ \frac{\cos\frac{1}{2}(\alpha-\gamma)}{\sin\frac{1}{2}(\beta+\delta)} = \frac{\cos\frac{1}{2}(\beta-\delta)}{\sin\frac{1}{2}(\alpha+\gamma)} = \left\{\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\right\}^{-1} \ \ldots\ldots \ (1).$$

Now $\cos\nu-\cos\mu = \cos(\gamma+\delta)-\cos(\alpha+\beta)$

$$= 2\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\sin\left\{\tfrac{1}{2}(\alpha-\gamma)+\tfrac{1}{2}(\beta-\delta)\right\}$$

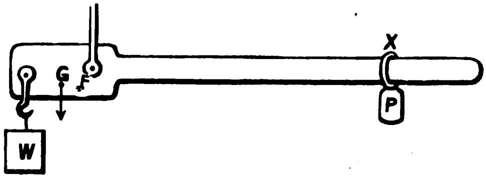
$$= 2\sin\tfrac{1}{2}(\alpha-\gamma)\sin\tfrac{1}{2}(\alpha+\gamma)+2\sin\tfrac{1}{2}(\beta-\delta)\sin\tfrac{1}{2}(\beta+\delta) \quad [\text{by } (1)],$$

$$= \cos\gamma-\cos\alpha+\cos\delta-\cos\beta;$$

$$\therefore\ \cos\mu+\cos\gamma+\cos\delta = \cos\nu+\cos\alpha+\cos\beta.$$

11850. (BELLE EASTON, B.Sc.)—The weight of a common steelyard is Q, and the distance of its fulcrum from the point from which the weight hangs is a when the instrument is in perfect adjustment. The fulcrum is displaced to a distance $a+\alpha$ from this end; show that the correction to be applied to give the true weight of a body, which in the imperfect instrument appears to weigh W, is $(W+P+Q)\left\{\alpha/(a+\alpha)\right\}$, P being the movable weight.

Solution by Rev. T. R. Terry, M.A.; W. J. Constable, M.A.; *and others.*



Let G be the centre of mass of the instrument, and F the fulcrum.

Let X be the position of P in the correct instrument when the real weight is W, then $\text{P}.\text{FX} = \text{Q}.\text{FG} + a.\text{W}$ .... (1).

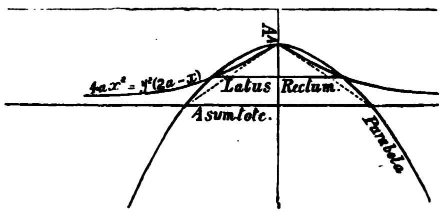
In the faulty instrument the weight of the substance is really $\text{W} - w$. But since W is the indicated weight, P remains at X. Therefore

$$\text{P}(\text{FX} - \alpha) = \text{Q}(\text{FG} + \alpha) + (a + \alpha)(\text{W} - w) \text{ .... (2).}$$

From (1) and (2) $\quad (\text{P} + \text{Q} + \text{W})\,\alpha = (a + \alpha)\,w.$

11631. (The Editor.)—Find the equation to the curve traced out in the same manner as the Cissoid of Diocles, when a parabola and its latus rectum are substituted in place of the generating circle and its diameter.

Solution by H. J. Woodall, A.R.C.S.; Professor Zerr; *and others.*



(1) When the ordinates are equidistant from the latus rectum.

Let $y^2 = 4ax$ be equation to parabola (P). Ordinate $x = k$ meets P at

$$y = \pm 2\,(ak)^{\frac{1}{2}}.$$

Lines through (0, 0) to this point

$$\left[k, \ \pm 2\,(ak)^{\frac{1}{2}}\right]$$

are $4ax^2 = y^2k$; these meet $x = 2a - k$ (equidistant with $x = k$ from latus rectum) and give the locus $4ax^2 = y^2(2a - x)$, a cubic.

(2) If the ordinates are chosen so that the tangents at extremities meet on latus rectum, we get another interesting curve.

Tangents at $(a/m^2, 2a/m)$, $(a/n^2, 2a/n)$ meet on the latus rectum if $mn = 1$.

Ordinate $x = a/m^2$ meets P at $y = \pm 2a/m$; lines through vertex to these points are $4m^2x^2 = y^2$, which meet $x = am^2$, if $4x^3 = ay^2$, a semicubical parabola.

9637. (R. Tucker, M.A.)—AD, BE, CF are the altitudes of the triangle ABC; k_1, k_1'; k_2, k_2'; k_3, k_3' are the S. points of the triangles

EAB, FCA; FBC, DAB; DCA, EBC respectively; prov. that

$$k_3'k_1 = k_1'k_2 = k_2'k_3 = \mathrm{R}\sin\mathrm{A}\sin\mathrm{B}\sin\mathrm{C}.$$

ρ_1, ρ_1'; ρ_2, ρ_2'; ρ_3, ρ_3' are the Brocard radii of the above triangles; prove that

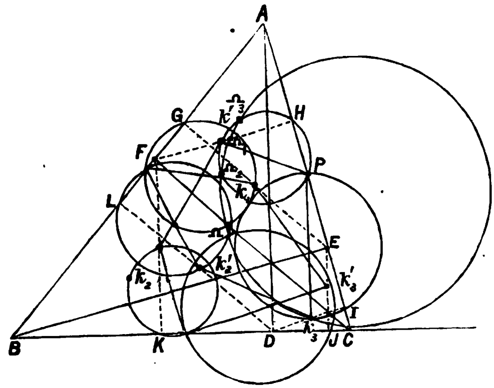
$$(1)\ \rho_1\rho_2\rho_3 = \rho_1'\rho_2'\rho_3';$$

$$(2)\ (\rho_2'^2-\rho_3^2)/a^2+(\rho_3'^2-\rho_1^2)/b^2+(\rho_1'^2-\rho_2^2)/c^2 = \tfrac{3}{64};$$

(3) the sets of four Brocard-points for the above pairs of triangles are concyclic (on three circles); (4) the tangent from any one of the right angles of the above triangles to the Brocard circle of the triangle is a mean proportional between the tangents to the same circle from the remaining (two) angles.

Solution by Professors ZERR, BHATTHACHARYA, *and others.*

The S. point of each triangle is found by drawing from its right angle a



line perpendicular to its opposite side, and taking its mid-point. We have

$$\mathrm{BG}\,.\,\mathrm{JE}+\mathrm{BJ}\,.\,\mathrm{GE} = \mathrm{GJ}\,.\,\mathrm{BE},\quad \mathrm{JE}\sin\mathrm{A}+\mathrm{GE}\sin\mathrm{C} = \mathrm{GJ},$$

$$\sin\mathrm{A}\sin\mathrm{C}\,(a\cos\mathrm{C}+c\cos\mathrm{A}) = \mathrm{GJ},$$

$$b\sin\mathrm{A}\sin\mathrm{C} = \mathrm{GJ},\quad c\sin\mathrm{A}\sin\mathrm{B} = \mathrm{KH},\quad a\sin\mathrm{B}\sin\mathrm{C} = \mathrm{LI};$$

$$\therefore\ \mathrm{GJ} = \mathrm{KH} = \mathrm{LI} = 2\mathrm{R}\sin\mathrm{A}\sin\mathrm{B}\sin\mathrm{C};$$

$$\therefore\ k_3'k_1 = \tfrac{1}{2}\mathrm{GJ} = k_1'k_2 = \tfrac{1}{2}\mathrm{HK} = k_2'k_3 = \tfrac{1}{2}\mathrm{LI} = \mathrm{R}\sin\mathrm{A}\sin\mathrm{B}\sin\mathrm{C}.$$

(1) Let $\omega_1, \omega_1', \omega_2, \omega_2', \omega_3, \omega_3'$ be the Brocard-angles; then $\omega_1 = \omega_1'$, $\omega_2 = \omega_2'$, $\omega_3 = \omega_3'$, for $\cot\omega_1 = \cot\omega_1' = \cot\mathrm{A}+\tan\mathrm{A}$;

$$\therefore\ b\rho_1 = c\rho_1' = \tfrac{1}{4}bc\,(1-3\cos^2\mathrm{A}\sin^2\mathrm{A})^{\frac{1}{2}},$$

$$c\rho_2 = a\rho_2' = \tfrac{1}{4}ac\,(1-3\cos^2\mathrm{B}\sin^2\mathrm{B})^{\frac{1}{2}},\quad a\rho_3 = b\rho_3' = \tfrac{1}{4}ab\,(1-3\cos^2\mathrm{C}\sin^2\mathrm{C})^{\frac{1}{2}};$$

$$\therefore\ \rho_1\rho_2\rho_3 = \rho_1'\rho_2'\rho_3',\quad (\rho_2'^2-\rho_3^2)\,a^2+(\rho_3'^2-\rho_1^2)\,b^2+(\rho_1'^2-\rho_2^2)\,c^2 = 0.$$

$$(2)\ (\rho_2'^2-\rho_3^2)/a^2+(\rho_3'^2-\rho_1^2)/b^2+(\rho_1'^2-\rho_2^2)/c^2$$
$$= \tfrac{1}{16}\Sigma\left(\frac{b^2}{c^2}-\frac{c^2}{b^2}\right)-\tfrac{3}{16}\Sigma\left(\frac{b^2}{c^2}-\frac{c^2}{b^2}\right)\sin^2\mathrm{A}\cos^2\mathrm{A}.$$

(3) Let Ω, Ω_1 be the Brocard-points of EAB; Ω_2, Ω_3 those of FCA. Since $\omega_1 = \omega_1'$, $\mathrm{A}\Omega_3\Omega$ and $\mathrm{A}\Omega_1\Omega_2$ are three each on a straight line.

But $A\Omega = c \sin\omega_1 / \sin A$, $A\Omega_3 = b \sin\omega_1 \cot A$, $A\Omega_2 = b\sin\omega_1/\sin A$, $A\Omega_1 = c\sin\omega_1 \cot A$;

$$\therefore\ A\Omega . A\Omega_3 = A\Omega_1 . A\Omega_2 \quad\text{or}\quad A\Omega : A\Omega_2 :: A\Omega_1 : A\Omega_3;$$

therefore Ω, Ω_1, Ω_2, Ω_3 are concyclic. Similarly for the remaining pairs of triangles.

(4) Let t, t_1, t_2 be the tangents from D, C, A to the Brocard circle. Then $t = (\mathrm{DI}.\mathrm{D}k_3)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} b \sin C \cos C$, $t_1 = (\mathrm{CP}.\mathrm{CI})^{\frac{1}{2}} = \frac{1}{\sqrt{2}} b \cos C$,

$$t_2 = (\mathrm{AP}.\mathrm{AI})^{\frac{1}{2}} = \frac{1}{\sqrt{2}} b \sin C; \qquad \therefore\ t^2 = t_1 t_2 \quad\text{or}\quad t_1 : t = t : t_2.$$

Similarly for the other triangles.

10631. (Professor Curtis, M.A. Suggested by 10497.)—If $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ are three conics having two common points P, Q, the equation of any conic passing through the same two points and touching the three conics is $\{(23)\,S_1\}^{\frac{1}{2}} \pm \{(31)\,S_2\}^{\frac{1}{2}} \pm \{(12)\,S_3\}^{\frac{1}{2}} = 0$, where (23) is found thus:—A common tangent is drawn to S_2 and S_3. The points of contact are joined to P and Q, and the area of the triangle formed by the tangent and the two joining lines is divided by the product of the three perpendiculars dropped from the three vertices to the line PQ. The quotient is (23).

Solution by the Proposer.

By Casey's *Conics*, the transformation

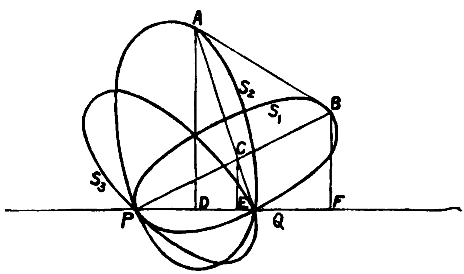
$$X = \frac{ax}{c+x}, \quad Y = \frac{ay}{c+x}$$

is equivalent to Geometrical, where the line at infinity is projected into $c + x = 0$. Therefore

$$\begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} = \frac{a^2 c}{(c+x_1)(c+x_2)(c+x_3)} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix};$$

or else, if the area of the triangle ABC be projected into A′B′C′ and p_1, p_2, p_3 be the perpendiculars from A′, B′, and C′, on the projection of the line and infinity,

$$ABC = \frac{a^2 . c}{p_1 . p_2 . p_3} A'B'C'.$$



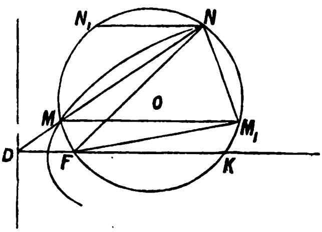
By Question 10444, the equation of a rectangular hyperbola touching three other rectangular hyperbolas, all four having parallel axes, is

$$(12\,.\,S_3)^{\frac{1}{2}} \pm (23\,.\,S_1)^{\frac{1}{2}} \pm (31\,.\,S_1)^{\frac{1}{2}} = 0,$$

where $12 =$ the area of the triangle formed by the common tangent to S_1 and S_2 and the lines drawn from the points of contact parallel to the asymptotes. When projected this becomes the equation of a conic touching three other conics, all four passing through two common points. In this case, let AB be the common tangent to S_1 and S_2; P and Q the common points of the conics S_1, S_2, S_3; AD, BF, CE perpendiculars on PQ from the vertices of the triangle ABC; 12 becomes $= \dfrac{\text{area ABC}}{\text{AD}\,.\,\text{BF}\,.\,\text{CE}}$; 23 and 31 having similar values.

11804. (R. Knowles, B.A.)—In Quest. 11727, if the circle FMN, of centre O, meet the axis again in K, prove that (1) OK bisects MN, (2) the locus of O is a semi-cubical parabola.

Solution by A. St. Clair; H. W. Curjel, B.A.; *and others.*



(1) In 11727 it is shown that

$$FM_1 = FN_1;$$

$\therefore$ arc $FM_1 =$ arc $FN_1 =$ arc NK;

but arc $FM_1 =$ arc MK;

$\therefore$ arc MK $=$ arc NK;

$\therefore$ OK bisects MN.

(2) Let equations to the parabola and DMN be

$$y^2 = 4a(x-a) \quad \text{and} \quad y = mx.$$

Equation to OK is

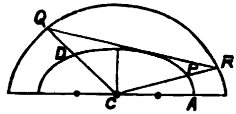
$$m(y-2a/m) + x - 2a/m^2 = 0; \quad \therefore \ DK = 2a + 2a/m^2;$$

$\therefore$ O is given by $x = 2a + a/m^2$, $m(y-2a/m) + x - 2a/m^2 = 0$, or $y = a/m^3$;

$\therefore$ equation to locus of O is $(x-2a)^3 = y^2a$, a semicubical parabola with cusp at focus of the parabola.

11889. (Professor Clifford, F.R.S.)—A tangent to an ellipse is a chord of a concentric circle, whose radius is equal to the distance between the ends of the axes of the ellipse; show that the straight lines which join the ends of the chord to the centre are conjugate diameters.

Solution by R. Chartres; Rev. J. J. Milne, M.A.; *and others*.



The equation to the tangent QR is

$$y = mx + (a^2m^2 + b^2)^{\frac{1}{2}}.$$

Making this homogeneous by

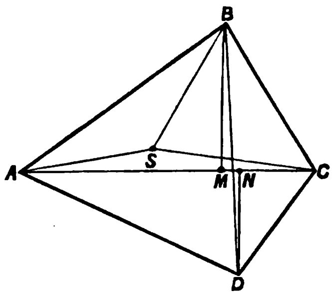
$$x^2 + y^2 = a^2 + b^2,$$

we get the equation to CR and CQ, and the product of the two values of $\left(\frac{y}{x}\right) = -\frac{b^2}{a^2}$, the condition for conjugate diameters.

[The circle is the director circle of the ellipse; hence any such chord QR is one side of a rectangle circumscribing the ellipse, and this is an orthogonal projection of a rhombus circumscribing a circle; therefore the diagonals of the rectangle, being projections of diameters at right angles, are conjugate diameters of the ellipse.]

11194. (R. Chartres.)—Give a simple proof, without infinitesimal changes, that, when four straight lines are given, the area enclosed will be a maximum when the figure is concyclic.

Solution by R. F. Davis, M.A.



Let ABCD be a convex quadrilateral, whose known sides AB, BC, CD, DA, are denoted by a, b, c, d, respectively. Let fall perpendiculars BM, DN on AC, then

$$a^2 - b^2 = AM^2 - CM^2$$
$$= AC\,(2AM - AC), \text{ \&c.};$$

$$\therefore\quad MN\,.\,AC = \tfrac{1}{2}\left\{(a^2+c^2) \sim (b^2+d^2)\right\} = \text{constant}.$$

But $AC^2(BM+DN)^2 + AC^2.MN^2 = AC^2\,.\,BD^2.$

Hence, area of quadrilateral is greatest when AC . BD is greatest.

Describe upon AB a triangle ABS directly similar to DBC; then the triangles ABD, SBC are also directly similar, and

$$ac + bd = BD\,(AS + BS) > AC\,.\,BD.$$

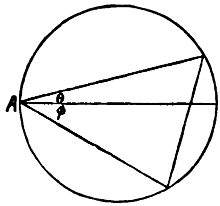
Hence AC . BD is greatest when S lies on AC, and then

$$\text{angles } ASB + BSC = C + A = 180°.$$

[On comparison it will be seen that the above proof is little more than a geometrical view of the Proposer's elegant and concise analytical solution given in Vol. 57, p. 35.]

11760. (Professor MUKHOPADHYAY, M.A.)—Prove that the mean value of the area of all (1) the acute-angled triangles inscribed in a given circle of radius a is $3a^2/\pi$, and of all (2) the obtuse-angled triangles is a^2/π.

Solution by H. W. CURJEL, B.A.; Prof. ZERR, M.A.; *and others.*



Let θ and ϕ be the angles which the two sides of the triangle through A make with the diameter through A.

(1) Then mean value

$$= 2a^2 \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi-\theta} \frac{\cos\theta\cos\phi\sin(\theta+\phi)\,d\theta\,d\phi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\theta} d\theta\,d\phi}$$

$$= \frac{16a^2}{\pi^2}\int_0^{\frac{1}{2}\pi}\left\{\cos\theta\sin\theta\left(\frac{\pi}{4}-\frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2}\right)+\tfrac{1}{2}\cos^4\theta\right\}d\theta = \frac{3a^2}{\pi}.$$

(2) Let the obtuse angle be at A. Then mean value

$$= \left\{2a^2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\cos\theta\cos\phi\sin(\theta+\phi)\,d\theta\,d\phi - \frac{3a^2}{8}\pi\right\}\Big/\left\{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}d\theta\,d\phi-\frac{\pi^2}{8}\right\}$$

$$= \left(\frac{\pi a^2}{2}-\frac{3a^2\pi}{8}\right)\Big/\left(\frac{\pi^2}{4}-\frac{\pi^2}{8}\right) = \frac{a^2}{\pi}.$$

11667. (Ven. Archdeacon WILSON, M.A.)—When $4n+1$ is a prime number, it is an old property of numbers that it is expressible in the form of the sum of two squares. But the proofs throw little or no light on the "reason why." Can any connexion be shown, or any explanation given, of this curious property?

Solution by ROBERT RAWSON.

Every number is represented by $(2n)$ or $(2p-1)$, (n) and (p) being integers. If, therefore, a prime number is expressed by the sum of two squares, one must be even and the other odd. Hence

$$(2n)^2+(2p-1)^2 = 4(n^2+p^2-p)+1 = 4m+1, \text{ if } m = n^2+p^2-p \dots (1, 2).$$

(1) shows that $(4m+1)$ is the sum of two squares when (2) is satisfied, whether $(4m+1)$ is a prime number or otherwise.

It may be urged that (1) and (2) do not clearly show that their solutions include all prime numbers of the form $(4m+1)$; still, it would surprise me to find a prime number, of the form specified, which is not included in (1) and (2) by giving suitable values to (n) and (p). There are, however, surprises in the theory of numbers, as in other places; for instance, FERMAT asserted that (2^a+1) is a prime number when (a) is any term in the series 1, 2, 4, 8, 16, &c.; but EULER found that

$$2^{32}+1 = 641 \times 6700417,$$

which is not a prime number

The following method of proof is elementary and free from the objection urged against the conclusion that the solutions of (1) and (2) include all prime numbers of the form $(4m+1)$. It depends upon Wilson's and Euler's theorems.

If $(4n+1)$ is a prime number, it must therefore satisfy Wilson's theorem. Then $1.2.3.4 \dots 4n+1 = (4n+1) M$ .... (3),

which takes the form $(1.2.3.4 \dots 2n)^2 + 1 = (4n+1) N$ .... (4).

(4) shows that the sum of two squares, prime to each other, has a divisor $(4n+1)$.

Euler has proved that every divisor of the sum of two squares prime to each other is also the sum of two squares. See Euler's *Alg.*, pp. 389–391, Barlow's *Theory of Numbers*, p. 194.

Hence it follows that $(4n+1)$ in (4) is the sum of two squares.

I have not had the pleasure of seeing the proofs of this curious property of numbers referred to by the Ven. Archdeacon Wilson. They must, however, differ from the one here given, which clearly shows the "reason why," through the channels of Wilson's and Euler's well-established theorems.

The proofs of Wilson's and Fermat's theorems from Lagrange may be interesting, as they are but little known to the English reader. They have not found a local habitation in the text-books of this country, although they have been mentioned in glowing terms by the late H. J. S. Smith, M.A., in his Report on the Theory of Numbers to the British Association for the Advancement of Science. Mr. Smith, a very great authority on the theory of numbers, designates Lagrange's proof of Fermat's theorem as *remarkable*, and implies his regret that it should be so little known in this country as to escape the attention of elementary writers on mathematical subjects. The principle depends upon the simple product of $(p-1)$ binomial factors, viz.,

$$(x+1)(x+2)(x+3) \dots (x+p-1).$$

Let $(x+1)(x+2)\dots(x+p-1) = x^{p-1} + pA_1x^{p-2} + pA_2x^{p-3} + \dots pA_{p-2}x + A_{p-1}$ .... (5),

where p is a prime number and x any number not a multiple of (p). From (1) $A_1, A_2, \dots A_{p-1}$ can be determined in functions of (p) as follows:—

In (1), put $x+1$ for x, and multiply the result by $x+1$; then

$$(x+1)(x+2)\dots(x+p) = (x+1)^p + pA_1(x+1)^{p-1} + pA_2(x+1)^{p-2} + \dots pA_{p-1} \quad \text{......... (6).}$$

Multiply (1) by $x+p$, then it becomes

$$(x+1)(x+2)\dots(x+p) = x^p + p(1+A_1)x^{p-1} + p(pA_1+A_2)x^{p-2}$$
$$+ p(pA_2+A_3)x^{p-3} + \dots pA_{p-1} \quad \text{...... (7).}$$

(6) and (7) are identical, and, by equating the coefficients of the same powers of x, the values of $(A_1, A_2, \dots A_{p-1})$ are readily found.

Of course each term on the right-hand side of (6) must be developed by

the binomial theorem. Comparing this result with the right-hand side of (7), it follows that

$$A_1 = \frac{p-1}{2}$$

$$2A_2 = \frac{p-1\,.\,p-2}{1.2.3} + \frac{p-1\,.\,p-2}{1.2}A_1$$

$$3A_3 = \frac{p-1\,.\,p-2\,.\,p-3}{1.2.3.4} + \frac{p-1\,.\,p-2\,.\,p-3}{1.2.3}A_1 + \frac{p-2\,.\,p-3}{1.2}A_2$$

$$4A_4 = \frac{p-1\,.\,p-2\,.\,p-3\,.\,p-4}{1.2.3.4.5} + \frac{p-1\,.\,p-2\,.\,p-3\,.\,p-4}{1.2.3.4}A_1 + \frac{p-2\,.\,p-3\,.\,p-4}{1.2.3}A_2 + \frac{p-3\,.\,p-4}{1.2}A_3$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

&c. &c. &c.

The value of rA_r can be written at sight.

In (5), put $x = 0$, then $A_{p-1} = 1.2.3 \dots p-1$ .... (8).

In (5), put $x = 1$, then $1.2.3 \dots p = 1 + pM + A_{p-1}$ .... (9).

From (8) and (9), $1.2.3 \dots (p-1)+1 = p\left\{1.2.3 \dots (p-1) - M\right\}$ ... (10).

(10) is the expression of Wilson's theorem. From (5),

$$x+1\,.\,x+2 \dots x+p-1 = x^{p-1} + pQ + A_{p-1}$$

$$= x^{p-1} + pQ + p\left\{1.2.3 \dots (p-1) - M\right\} - 1, \text{ from (9)};$$

$$\therefore\ x^{p-1} - 1 = x+1\,.\,x+2 \dots (x+p-1) - p\left\{Q - M + 1.2.3 \dots (p-1)\right\} \dots (11).$$

(11) expresses Fermat's theorem, observing that (x) may be any integer which is not a multiple of (p); when this condition is fulfilled, then $x+1\,.\,x+2 \dots x+p-1$ is obviously a multiple of (p).

There seems to be but little known of the history of Sir John Wilson or the method by which he reached his theorem. Perhaps he induced its generality from particular cases and not from any elaborate theory of prime numbers. However this may be, his theorem has established his title as an original mathematician of eminence. Both Euler and Lagrange considered the proof of it to be difficult.

The demonstrations of Peacock and Barlow do not contrast favourably with the one here given from Lagrange in 1771, either in force or simplicity.

[For other Solutions, see Vol. lviii., p. 113, and Vol. lix., p. 84.]

11503. (W. J. Greenstreet, M.A.) — In a right-angled triangle ABC, draw Bl perpendicular to the hypotenuse AC, lm perpendicular to AB, mn perpendicular to AC, np perpendicular to AB, and so on. Find (1) the sum of these perpendiculars in terms of the sides of the triangle. From a point P in AC, a perpendicular PQ is let fall on AB; if $PQ^2 = AP\,.\,PC$, (2) find P. Draw BD and CE perpendicular to the bisector AZ of the angle A; show that (3) the middle point of BC, B, D, E are concyclic, and (4) the area of the triangle BDE is equal to BD . AE.

Solution by Professors Krishnamarcham, Aiyar, *and others.*

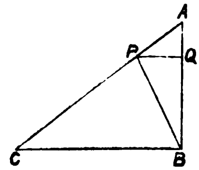
1. We have $\quad BL = c \sin A = \frac{ac}{b}, \quad Lm = BL \cos A = \frac{ac^2}{b^2},$

$$mn = Lm \cos A = \frac{ac^3}{b^3}, \text{ and so on;}$$

hence the sum required

$$= \frac{ac}{b}\left(1 + \frac{c}{b} + \frac{c^2}{b^2} + \dots \text{ to } \infty\right) = \frac{ac}{b-c} = \frac{ac\,(b+c)}{b^2-c^2} = \frac{c}{a}(b+c).$$

2. We have $\quad PQ = a\,\frac{AP}{AC};$

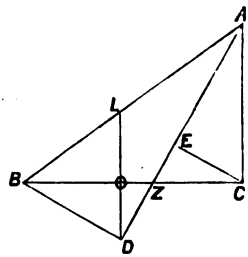


therefore $\quad AP \,.\, PC = a^2\,\frac{AP^2}{AC^2},$

or $\quad \frac{AP}{PC} = \frac{b^2}{a^2};$

therefore $\quad AP = \frac{b^2}{a^2+b^2} \,.\, AC = \frac{b^3}{a^2+b^2}.$

3. Draw DO perpendicular to BC, and produce to meet BA in L.



Then $\angle$ LDA = DAC = LAD (because LD is parallel to AC); therefore

$$LD = LA,$$

and $\quad \angle BDL = 90^\circ - \frac{1}{2}A = ABD;$

therefore $\quad BL = LD = LA;$

i.e., L is the middle point of AB. But LD is parallel to AC. Therefore O is the mid-point of BC, and, DOC, DEC being each = 90°, D, O, E, C are concyclic.

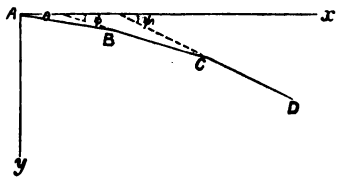
This latter section of the question may be generalized:—If ABC be *any* triangle, and AK the perpendicular on the base, CE, BD perpendiculars on the bisector of A, then K, E, O, D are concyclic, where O is the middle point of the base.

4. $\quad BD \,.\, AE = c \sin \frac{1}{2}A \,.\, b \cos \frac{1}{2}A = \frac{1}{2}bc \sin A = \Delta ABC.$

Also the triangles BDA, CEA are similar; therefore $\Delta ABC = CE \,.\, AD.$

11223. (Professor Mukhopadhyay.)—AB, BC, CD are three equal uniform rods freely jointed together and movable about the extremity A; the rods fall from a horizontal position of rest; prove that (1) the radius of curvature of the initial path of the extremity D of the further rod is $\frac{81}{131}a$, where a is the length of each rod; and (2) the initial stresses at C, B are in the ratio of 1 : 4 : 15.

Solution by H. W. CURJEL, B.A.



Take the axes of x and y horizontal and vertically downwards through A.

Let θ, ϕ, ψ be the inclinations of the rods to the horizon, and x_1, y_1; x_2, y_2; x_3, y_3 the coordinates of their centres of gravity, and x, y those of D.

Let X_1Y_1, X_2Y_2, X_3Y_3 be the components of the reactions at A, B, C.

The geometrical equations are $\quad x_1 = \frac{1}{2}a\cos\theta$,

$$y_1 = \tfrac{1}{2}a\sin\theta,\quad x_2 = \tfrac{1}{2}a(2\cos\theta+\cos\phi),\quad y_2 = \tfrac{1}{2}a(2\sin\theta+\sin\phi),$$

$$x_3 = \tfrac{1}{2}a(2\cos\theta+2\cos\phi+\cos\psi),\quad y_3 = \tfrac{1}{2}a(2\sin\theta+2\sin\phi+\sin\psi).$$

The equations of motion are

$$m\ddot{y}_1 = mg - Y_1 + Y_2,\quad m\ddot{y}_2 = mg - Y_2 + Y_3,\quad m\ddot{y}_3 = mg - Y_3 \;\ldots\ldots(1),$$

$$m\ddot{x}_1 = -X_1 + X_2,\qquad m\ddot{x}_2 = -X_2 + X_3,\qquad m\ddot{x}_3 = -X_3 \;\ldots\ldots\ldots(2),$$

$$\left.\begin{aligned} \tfrac{1}{12}(ma^2)\ddot{\theta} &= (Y_1+Y_2)\tfrac{1}{2}a\cos\theta - (X_1+X_2)\tfrac{1}{2}a\sin\theta \\ \tfrac{1}{12}(ma^2)\ddot{\phi} &= (Y_2+Y_3)\tfrac{1}{2}a\cos\phi - (X_2+X_3)\tfrac{1}{2}a\sin\phi \\ \tfrac{1}{12}(ma^2)\ddot{\psi} &= Y_3\tfrac{1}{2}a\cos\psi - X_3\tfrac{1}{2}a\sin\psi \end{aligned}\right\}\;\ldots\ldots\ldots\ldots(3).$$

Initially, $\theta = \phi = \psi = 0$; hence, differentiating the geometrical equations twice,

$$\ddot{x}_1 = \ddot{x}_2 = \ddot{x}_3 = 0 \text{ and } \ddot{y}_1 = \tfrac{1}{2}a\ddot{\theta},\quad \ddot{y}_2 = \tfrac{1}{2}a(2\ddot{\theta}+\ddot{\phi}),\quad \ddot{y}_3 = \tfrac{1}{2}a(2\ddot{\theta}+2\ddot{\phi}+\ddot{\psi}) \;\ldots(4).$$

These, with equations (2), give $\quad X_1 = X_2 = X_3 = 0$;

and, with equations (1) and (3), give

$$Y_1 + Y_2 = \tfrac{1}{6}(ma\ddot{\theta}) = \tfrac{1}{3}(mg - Y_1 + Y_2),$$

$$Y_2 + Y_3 = \tfrac{1}{6}(ma\ddot{\phi}) = \tfrac{1}{3}(-mg - 3Y_2 + 2Y_1 + Y_3),$$

$$Y_3 = \tfrac{1}{6}(ma\ddot{\psi}) = \tfrac{1}{3}(mg - 3Y_3 + 4Y_2 - 2Y_1).$$

Hence $\qquad Y_1 = \frac{15}{52}mg,\quad Y_2 = -\frac{1}{13}mg,\quad Y_3 = \frac{1}{52}mg$;

therefore $\; Y_1 : -Y_2 : Y_3 = 15 : 4 : 1$, and $\; \ddot{\theta} : \ddot{\phi} : \ddot{\psi} = 11 : -3 : 1$.

Again, $\; x = a(\cos\theta+\cos\phi+\cos\psi),\quad y = a(\sin\theta+\sin\phi+\sin\psi)$

Let $\rho =$ the initial radius of curvature of the path of D.

$$\text{Then}\quad \rho = \left(\frac{dy}{d\theta}\ddot{\theta} + \frac{dy}{d\phi}\ddot{\phi} + \frac{dy}{d\psi}\ddot{\psi}\right)^2 \Big/ \left(\frac{d^2x}{d\theta^2}\ddot{\theta}^2 + \frac{d^2x}{d\phi^2}\ddot{\phi}^2 + \frac{d^2x}{d\psi^2}\ddot{\psi}^2\right)$$

(Art. 464, ROUTH's *Elem. Rigid Dynam.*, 4th Edit.), since

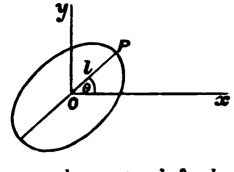
$$\frac{d^2x}{d\phi\,d\psi} = \frac{d^2x}{d\psi\,d\theta} = \frac{d^2x}{d\theta\,d\phi} = 0;\qquad \therefore\; \rho = \frac{a^2(11-3+1)}{-a(11^2+3^2+1)} = -\frac{81a}{131}.$$

11710. (W. J. JOHNSTONE.)—If $y = \lambda x$ is an axis of

$$ax^2 + 2hxy + by^2 + c' = 0,$$

prove that (1) its length is $2\left[-c'/(a+h\lambda)\right]^{\frac{1}{2}}$; (2) the equation referred to its axes is $$x^2(a+h\lambda)+y^2(a+h\lambda')+c'=0.$$

Solution by C. MORGAN, M.A.; H. W. CURJEL, B.A.; *and others.*



$$x^2=\frac{-c'}{a+2h\lambda+b\lambda^2}$$

gives the point P, where $y=\lambda x$ cuts the curve.

$$l^2=x^2\sec^2\theta=-\frac{c'}{a+2h\lambda+b\lambda^2}(1+\lambda^2)\ \dots\ (1)$$

l must be a maximum or minimum.

Hence $(a+2h\lambda+b\lambda^2)\,2\lambda-(1+\lambda^2)(2h+2b\lambda)=0$; $\therefore\ b\lambda=a\lambda+h\lambda^2-h.$

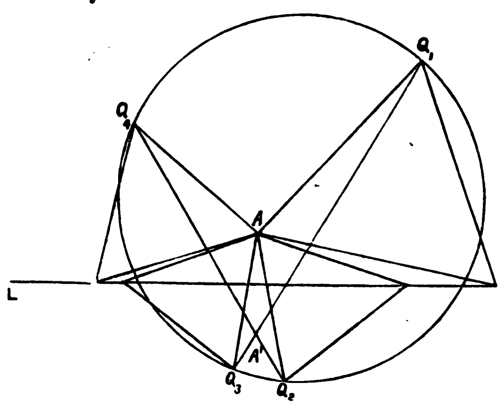
From (1) $$l^2=-\frac{c'}{a+h\lambda+a\lambda^2+h\lambda^3}(1+\lambda^2)=-\frac{c'}{a+h\lambda};$$

$\therefore$ equation to conic is $x^2(a+h\lambda)+y^2(a+h\lambda')+c'=0.$

11882. (A. KAHN, M.A.)—Construct an equilateral triangle, such that one vertex coincides with a given point, and the other two vertices are on a given straight line and a given circle, respectively.

Solution by W. J. DOBBS, B.A.; Professor SHIELDS; *and others.*

The following construction, giving four solutions, follows from the subjoined analysis:—



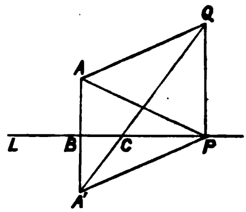
Draw two straight lines through A′, making 60° with the given straight line. Let these meet the given circle in the points Q_1, Q_2, Q_3, Q_4. Then AQ_1, AQ_2, AQ_3, AQ_4 are each sides of the required triangles.

Analysis.—Let A be the given point, L the given straight line.

Draw ABA' perpendicular to L, making
$$BA' = AB.$$

Take any point P in L, and on AP describe an equilateral triangle APQ, and let A'Q meet L in C.

Then the circle with centre P and radius PA passes through A', A, Q, and therefore
$$AQA' = \tfrac{1}{2}APA' = APB.$$

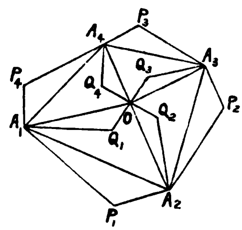


Therefore Q, A, C, P are concyclic. Therefore $QCP = 60°$. Therefore locus of all such points Q is two straight lines through A' making 60° with L.

11717. (Professor RAMASWAMI AIYAR.) — If similar triangles be described on the sides of a polygon in order, prove that the centre of gravity of equal particles placed at their vertices will coincide with that of equal particles placed at the vertices of the polygon.

Solution by the PROPOSER.

Let A_1, A_2, A_3, A_4 be the polygon; P_1, P_2, P_3, P_4 the vertices of the triangles. Take any point O within the polygon; join $A_1O \dots A_4O$, and on them describe triangles similar to those described on the sides.



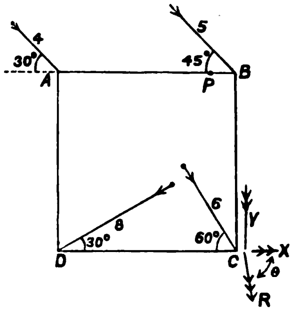
Now, suppose equal particles placed at $A_1 \dots A_4$, and one such particle at Q_1. Since $P_1Q_1Q_2A_2$ is a square, the particles at Q_1, A_2 may be transferred to P_1, Q_2 without affecting the C.G. of the system; and, since $P_2Q_2Q_3A_3$ is a square, the particles at Q_2, A_3 may be transferred to P_2, Q_3; similarly, the particles Q_3, A_4 to P_3, Q_4; and lastly Q_4, A_1 to P_4, Q_1.

Thus the particle originally at Q_1 is again at Q_1; but (A_1, A_2, A_3, A_4) have been transferred to (P_1, P_2, P_3, P_4) without affecting the C.G. of the system. This proves the theorem.

11634. (I. ARNOLD.)—ABCD is a rigid body in the form of a square, whose base AB is 10 inches. Four forces, proportional to 4, 5, 6, and 8, act in the plane of the square at the angular points A, B, C, D, making with the direction AB the angles 30°, 45°, 60°, and 150° respectively; required the magnitude, direction, and point of application of a force which, acting on AB, shall keep the square in equilibrium.

Solution by H. J. WOODALL, A.R.C.S.

Resolve the forces into X, Y, parallel to AB, BC; then we have



$$X = 4 \times \tfrac{1}{2} \times \sqrt{3} + 5 \times \tfrac{1}{2} \times \sqrt{2} + 6 \times \tfrac{1}{2} - 8 \times \tfrac{1}{2} \times \sqrt{3}$$
$$= -2\sqrt{3} + 2\tfrac{1}{2}\sqrt{2} + 3$$
$$= 3{\cdot}0714324.$$
$$Y = 4 \times \tfrac{1}{2} + 5 \times \tfrac{1}{2} \times \sqrt{2} + 6 \times \tfrac{1}{2} \times \sqrt{3} + 8 \times \tfrac{1}{2}$$
$$= 6 + 2\tfrac{1}{2}\sqrt{2} + 3\sqrt{3}$$
$$= 14{\cdot}7316864.$$
$$R = (X^2 + Y^2)^{\frac{1}{2}} = 15{\cdot}0484;$$
$$\theta = \tan^{-1}(Y/X) = \tan^{-1} 4{\cdot}796357 = 78^\circ\, 13\tfrac{1}{2}'.$$

To find point of application in AB. Take $AP = x$; find moments about P. We get $x(8 + 2\tfrac{1}{2}\sqrt{2} + 3\sqrt{3}) = 10(2\tfrac{1}{2}\sqrt{2} + 7\sqrt{3} - 3)$;

$$\therefore\ x \times 14{\cdot}61037 = 126{\cdot}59890; \qquad \therefore\ x = 8{\cdot}665.$$

11803. (E. A. WHITE, B.A.)—Solve the system of equations

$$\frac{dy_1}{dx} = y_2 - y_1^2, \quad \frac{dy_2}{dx} = y_3 - y_1 y_2, \ \dots \ \frac{dy_{n-1}}{dx} = y_n - y_1 y_{n-1}, \quad \frac{dy_n}{dx} = 1 - y_1 y_n,$$

and show that, in the case when $n = 2$, a particular solution is

$$y_1 = t(x), \quad y_2 = T(x),$$

where the functions satisfy $\quad t(u+v) = \dfrac{t(u) + t(v) + T(u) \,.\, T(v)}{1 + t(u) \,.\, T(v) + T(u) \,.\, t(v)},$

and a similar relation got by interchanging t and T.

Solution by the PROPOSER.

Put $y_1 = \dfrac{1}{u}\dfrac{du}{dx}$. Substitute in (1), then $y_2 = \dfrac{1}{u}\dfrac{d^2u}{dx^2}$; in the second, then

$$y_3 = \frac{1}{u}\frac{d^3u}{dx^3}, \ \&\text{c., and} \quad y_n = \frac{1}{u}\frac{d^n u}{dx^n}.$$

Therefore, from the last, $\quad \dfrac{d^{n+1}u}{dx^{n+1}} - u = 0$;

and hence the solution of the system is at once found.

When $n = 2$, then $\quad u = Ae^x + Be^{\omega x} + Ce^{\omega^2 x}$.

Take $A = B = C = 1$, and let $f(x) = \tfrac{1}{3}(e^x + e^{\omega x} + e^{\omega^2 x})$;
by multiplication, it can be shown that

$$f(u+v) = f(u) \,.\, f(v) + f'(u) \,.\, f''(v) + f''(u) \,.\, f'(v) \ \dots\dots\dots (1);$$

$$\therefore\ \left.\begin{array}{l} f'(u+v) = f''(u).f''(v) + f(u).f'(v) + f'(u).f(v) \\ f''(u+v) = f'(u).f'(v) + f(u).f''(v) + f''(u).f(v) \end{array}\right\} \quad \text{and} \quad \ldots\ldots\ldots(2).$$

For $f'''(u) = f(u)$ put $t(u) = f'(u)/f(u)$, $\mathrm{T}(u) = f''(u)/f(u)$;

then $\left.\begin{array}{l} y_1 = t(u) \\ y_2 = \mathrm{T}(u) \end{array}\right\}$ is a solution, and $t(u+v) = \dfrac{t(u)+t(v)+\mathrm{T}(u).\mathrm{T}(v)}{1+t(u).\mathrm{T}(v)+\mathrm{T}(u).t(v)}$

From (1) and (2), &c.

11849. (Rev. T. R. Terry, M.A.)—Prove the identity

$$\begin{aligned} &(b-c)(b-d)(c-d)(x-b)(x-c)(x-d)(a^3+pa^2+qa+r) \\ &-(c-d)(c-a)(d-a)(x-c)(x-d)(x-a)(b^3+pb^2+qb+r) \\ &+(d-a)(d-b)(a-b)(x-d)(x-a)(x-b)(c^3+pc^2+qc+r) \\ &-(a-b)(a-c)(b-c)(x-a)(x-b)(x-c)(d^3+pd^2+qd+r) \\ = \ &(x^3+px^2+qx+r)(a-b)(a-c)(a-d)(b-c)(b-d)(c-d). \end{aligned}$$

Solution by the Proposer.

By the usual rule for "Partial Fractions," we have

$$\frac{x^3+px^2+qr+r}{(x-a)(x-b)(x-c)(x-d)} = \frac{a^3+pa^2+qa+r}{(x-a)(a-b)(a-c)(a-d)}$$

+ three similar terms.

Clearing of fractions, we get the required identity.

11263. (Professor Wolstenholme, Sc.D.)—Prove that (1) if $a^2<1$, $\int_0^{\frac{1}{2}\pi}(\tan x)^a\,dx = \frac{1}{2}\pi\sec\frac{1}{2}\pi a$; and thence (2) the coefficient of $\frac{x^n}{n!}$ in the expansion of $\sec x$ is $\left(\frac{2}{\pi}\right)^{n+1}\int_0^{\frac{1}{2}\pi}(\log\tan x)^n\,dx$; also, if $a^2<1$,

$$\int_{-\infty}^{\infty}\frac{\sinh ax}{\cosh x}\frac{dx}{x} = 2\log\tan\tfrac{1}{4}\pi(1+a), \qquad \int_{-\infty}^{\infty}\frac{\sin ax}{\cosh x}\frac{dx}{x} = 2\tan^{-1}(\sinh\tfrac{1}{2}\pi a).$$

Solution by H. W. Curjel, B.A.

$$\int_0^{\frac{1}{2}\pi}(\tan x)^a\,dx = \int_0^{\infty}\frac{y^a}{1+y^2}\,dy, \text{ putting } y=\tan x.$$

But
$$\int_0^{\infty}\frac{y^{s-1}}{1+y^r}\,dy = \frac{\pi}{r\sin(s/r)\pi},$$

where r and s are positive, and $s<r$ (Todh., *Int. Calc.*, Art. 255).

Therefore, putting $r=2$ and $s=1+a$ ($\because\ a^2<1$),

$$\int_0^{\infty}\frac{y^a}{1+y^2}\,dy = \frac{\pi}{2\sin\frac{1}{2}(a+1)\pi} = \tfrac{1}{2}\pi\sec\tfrac{1}{2}\pi a.$$

(2) The coefficient of $x^n/n!$, in the expansion of $\sec x$, is, by Maclaurin's

Theorem, $$= \left\{ \left(\frac{d}{dx}\right)^n \sec x \right\}_{x=0}$$

$$= \left(\frac{2}{\pi}\right)^{n+1} \left\{ \left(\frac{d}{da}\right)^n \tfrac{1}{2}\pi \sec \tfrac{1}{2}\pi a \right\}_{a=0}, \text{ putting } x = \tfrac{1}{2}\pi a$$

$$= \left(\frac{2}{\pi}\right)^{n+1} \int_0^{\frac{1}{2}\pi} \left\{ \left(\frac{d}{da}\right)^n (\tan x)^a \right\}_{a=0} dx = \left(\frac{2}{\pi}\right)^{n+1} \int_0^{\frac{1}{2}\pi} (\log \tan x)^n \, dx.$$

(3) Let $$X = \int_{-\infty}^{\infty} \frac{\sinh ax}{\cosh x} \frac{dx}{x}, \quad Y = \int_{-\infty}^{\infty} \frac{\sin ax}{\cosh x} \frac{dx}{x};$$

$$\frac{dX}{da} = \int_{-\infty}^{\infty} \frac{\cosh ax}{\cosh x} dx = \int_{-\infty}^{\infty} \frac{e^{ax} + e^{-ax}}{e^x + e^{-x}} dx$$

$$= \int_0^{\infty} \frac{y^a + y^{-a}}{1 + y^2} dy \text{ (putting } e^x = y) = \pi \sec \tfrac{1}{2}\pi a.$$

$$\therefore \quad X = 2 \log \tan \tfrac{1}{4}\pi (1 + a), \text{ for } X = 0 \text{ when } a = 0.$$

Also $$\frac{dY}{da} = \int_{-\infty}^{\infty} \frac{\cos ax}{\cosh x} dx$$

$$= \frac{\pi}{\cosh \frac{1}{2}\pi a} \left(\text{putting } a\sqrt{(-1)} \text{ for } a \text{ in the value of } \frac{dX}{da}\right).$$

$$\therefore \quad Y = \int_0^a \frac{\pi}{\cosh \frac{1}{2}\pi a} da = \int_0^a \frac{2\pi e^{\frac{1}{2}\pi a}}{1 + e^{\pi a}} da = 2 \int_1^{e^{\frac{1}{2}\pi a}} \frac{2dy}{1 + y^2} \text{ (putting } e^{\frac{1}{2}\pi a} = y).$$

$$= 4 \left\{ \tan^{-1} (e^{\frac{1}{2}\pi a}) - \tan^{-1} 1 \right\} = 4 \tan^{-1} \frac{e^{\frac{1}{2}\pi a} - 1}{1 + e^{\frac{1}{2}\pi a}} = 2 \tan^{-1} \left\{ \sinh \tfrac{1}{2}\pi a \right\}.$$

11658. (Professor Ramaswami Aiyar.)—From each of n equal straight lines is cut off a piece at random: the chance that the greatest of the pieces cut off exceeds the sum of all the others is $1 : (n-1)!$; and the chance that the square on the greatest exceeds the sum of the squares on all the others is $(\frac{1}{4}\pi)^{\frac{1}{2}(n+1)} : \Gamma\left\{\frac{1}{2}(n+1)\right\}$.

Solution by H. W. Curjel, B.A.; Professor Zerr; *and others.*

Let a equal length of each straight line. The chance that any particular piece cut off is greater than the sum of the rest

$$= \frac{\int_0^a \left[\int_0^{x_1} \left\{ \int^{x_1 - x_2} \left[\int_0^{x_1 - x_2 - x_3} (\dots) \, dx_4 \right] dx_3 \right\} dx_2 \right] dx_1}{\int_0^a dx_1 \int_0^a dx_2 \int_0^a dx_3 \int_0^a dx_4 \dots} = \frac{a^n (1/n!)}{a^n} = \frac{1}{n!}.$$

Hence the chance that the greatest is greater than the sum of the rest

$$= n/n! = 1/(n-1)!.$$

Chance that the square of the greatest is greater than the sum of the squares of the rest

$$= n \int_0^a (\text{volume of hypersphere of } n-1 \text{ dimensions and radius } x) \, dx / a^n$$

$$= (n/n)(1/2^{n-1}) \left\{ \Gamma(\tfrac{1}{2}) \right\}^{n-1} / \left[\Gamma\left\{ \tfrac{1}{2}(n-1) + 1 \right\} \right] = (\tfrac{1}{4}\pi)^{\frac{1}{2}(n-1)} / \left[\Gamma\left\{ \tfrac{1}{2}(n+1) \right\} \right].$$

APPENDIX.

UNSOLVED QUESTIONS.

1035. (Hibernicus.)—Place a chord in a given circle such that it will pass through a given point; and if perpendiculars be drawn from its extremities upon two right lines given in position, the rectangle or ratio of these shall be given.

1038. (S. Watson.)—A straight line always cuts off a given area from a given parabola; find the curve which it always touches.

1047. (Matthew Collins, B.A.)—Prove that any whole number whose first and last figures are alike, and all whose middle figures are 0, can never be exactly divisible by 31, 37, 41, 43, 53, 67, 71, 79, or 83.

1052. (Editor.)—Find two numbers, such that if from each of them and also from the sum of their squares, their product be subtracted, the three remainders shall be rational squares.

1076. (Editor.)—Show that in a perfectly mounted equatorial instrument the effect of refraction causes the star to describe a small curve in the field, which, when the zenith distance of the star is not very great, is a conic section whose position may be determined.

1080. (Editor.)—Three given weights (considered as heavy material points) are attached to the surface of a sphere; find the position of equilibrium of the sphere when resting on a horizontal plane, and give the result in the particular case in which the weights are arranged in a great circle.

1081. (Editor.)—Two weights are attached by short strings of given length to the surface of a cylinder; determine the position of equilibrium of the cylinder when resting on a horizontal plane, the weights resting upon the surface of the cylinder.

1090. (E. Harrison, M.A.)—In a right-angled triangle inscribe a square, one of its angles coinciding with the right angle of the triangle; in the two right-angled triangles thus formed inscribe in the same way squares; in the four right-angled triangles now formed inscribe as above squares; and continue this process; after the n^{th} operation there will be 2^n squares. It is required to show that the perimeters of these 2^n squares are equal to the perimeters of the first square.

1095. (Editor.)—A straight line intersects both loops of the lemniscate, one loop in P, Q, and the other in P′, Q′. Prove that the mid-points of PQ and of P′Q′ are equidistant from the centres of the curves.

1098. (Editor.)—Four spheres are drawn, so that every three have a common tangent line: show that these four tangents intersect in a point, and are equal in length as measured from the points of contact to the point of intersection.

1114. (Matthew Collins, B.A.)—Two persons, A and B, played altogether 77 games for one shilling per game, of which A won 35, and B the other 42. In how many different ways in this play was it possible for A to have been the clear winner of four shillings from B, before B was the clear winner of five shillings from A? [The solution of this question will supply a demonstration of one of the most curious and remarkable laws in the Doctrine of Chances given, but left undemonstrated, by De Moivre and Simpson. See De Moivre's *Doctrine of Chances*, problem 65, 3rd edition, page 211; and also Simpson's *Laws of Chance*, problems 25 and 26.]

1136. (Editor.)—Two tangents to a semi-cubical parabola include a *given* angle; required the locus of their point of intersection. What does the locus become when the *given* angle is a right angle?

1142. (Editor.)—Find how high above a given point on the earth's surface a person must be raised to see the sun at a given time in the night.

1157. (Sir James Cockle.)—Given $m+q=0,\quad n+mq+r=0,$

$$p+nq+mr=-5Q,\quad pq+nr=0,\quad pr=2Q\sqrt[5]{Q^2};$$

find m, n, p, q, r in terms of Q.

1158. (Mathematicus.) — Find the point at which a right-angled triangle must be fixed so that, when its right angle is struck by a blow perpendicular to its plane, it may begin to revolve about an axis parallel to its hypotenuse.

1164. (Stephen Watson.) — Find the area of the curve which is always touched by a line whereof a given portion is intercepted between two lines given in position.

1173. (Editor.)—In a given triangle place (1) a given circle, so that the area of the triangle polar to the given triangle, relatively to the given circle, may be a minimum; also, find (2) the locus of the centre of the circle when the area of the polar triangle is constant.

1185. (W. C. Otter, F.R.A.S.)—Suppose a man has a calf which at the end of three years begins to breed, and afterwards brings forth a female calf every year; and that each calf begins to breed in like manner at the end of three years, bringing forth a cow calf every year; and that these last breed in the same manner, &c.; find the owner's stock at the end of x years.

1187. (S. Watson.)—A circle is drawn at random, both in magnitude and position, but so as to lie wholly upon the surface of a given circle; find the chance that it will not exceed an n^{th} part of the given circle.

1195. (W. C. Otter, F.R.A.S.)—Find (1) by the properties of the cone, and (2) independently of the cone, the locus of the extremity of the shadow of a vertical gnomon erected on a horizontal plane on a given day in a given latitude.

1202. (Matthew Collins, B.A.)—Prove that, if ABC be the vertices of a hypocycloid of three branches described by a point on the circumference of a circle which rolls within a fixed circle whose diameter is three times

that of the rolling circle; and if A′BC be another equilateral triangle, A′ and A being upon opposite sides of BC, P being *any* point in the hypocycloid, PE perpendicular to PC, and PD a tangent to the circle whose centre is A′ and radius is A′B or A′C; then $PD^4 \div PE^3$ will be constant and equal to 16 times the diameter of the rolling circle.

1211. (S. Watson.)—Show that the *average* area of all the triangles that can be inscribed within a given triangle, is one-fourth of the triangle.

1215. (J. W. Mulcaster.)—Find the square which, when placed upon a sphere, and four planes drawn through its side and the centre of the sphere, shall cut off a given surface.

1233. (J. W. Mulcaster.)—Three rods are connected at their mid-points by strings of equal lengths, and thrown up; find the probability of their forming a triangle.

1238. (Editor.)—A conic passes through the angular points and the *centroid* of a given triangle; find the area of the locus of its centre.

1239. (Editor.)—Given one side of a right-angled triangle; construct it, so that the difference between the other side and the adjacent segment of the hypotenuse, cut off by a perpendicular from the right angle, may be a *maximum*. Prove that the perpendicular divides the hypotenuse in extreme and mean ratio, and that the greatest segment is equal to the remote side of the triangle.

1242. (T. T. Wilkinson, F.R.A.S.)—Given two right lines SA, SB, and three points O, P, Q, situated upon a third right line parallel to SB; draw any transversal through O cutting SA, SB, in a, b; join aP, bQ, intersecting each other in m; then the points m are all situated upon a right line given in position.

1254. (Editor.)—If the base BC of a triangle ABC be trisected in Q, R, prove that $\sin BAR \,.\, \sin CAQ = 4 \,.\, \sin BAQ \,.\, \sin CAR$,
and $(\cot BAQ + \cot QAR)(\cot CAR + \cot RAQ) = 4 \operatorname{cosec}^2 QAR$.

1267. (Editor.)—Find the probability that if n letters were put at random into n envelopes, already properly addressed, no letter would be put into the right envelope.

1268. (S. Watson.)—Two points A, B are taken at random upon the surface of a given circle, of centre O; find the chance that the circle through O, A, B shall be less than one-fourth of the given circle.

1276. (J. McDowell, B.A., F.R.A.S.)—Find the locus of a point such that, if parallels be drawn through it to the three sides of a triangle, the sum of the rectangles under the three pairs of intercepts on each line respectively, between the point and the two sides which it meets, shall be equal to a given rectangle. What is the position of the point *within* the triangle when the sum of the three rectangles is a *maximum*?

1280. (Editor.)—A *given* angle revolves round its vertex, which is fixed at the focus of a conic; and a tangent is drawn to the conic, at the point where it is cut by *one* of the sides of the angle; find the locus of the point in which this tangent meets *the other* side of the angle.

1295. (Editor.)—Find the locus of a point whose distance from *one* of three given points is (1) an *arithmetic* mean, (2) a *geometric* mean, between its distances from *the other two points*.

1309. (Editor.) — A rifleman stands exactly opposite the rectangular front of a house, the base of which is on a level with the lock of his rifle. Supposing it quite certain that he never depresses his rifle more than a given angle below the horizon, that all other directions are equally probable, and that the bullet flies sensibly in a straight line: determine the probability that, if the rifle go off accidentally, the bullet will strike the house.

1312. (Dr. Rutherford, F.R.A.S.) — A uniform beam rests with its ends against the opposite sides of an oblique shaft inclined to the vertical at an angle α; find the least coefficient of friction consistent with equilibrium, when the beam is of such a length as to be inclined to the vertical at an angle β.

1313. (Dr. Rutherford, F.R.A.S.) — A heavy rectangular beam, whose weight is W and length $2a$ feet, rests with one side of its square end, which is $2b$ feet, on a rough horizontal floor, and the corresponding side of the other square end against a rough vertical wall; find its position when it is on the point of slipping along the floor, and its pressure against the wall, μ being the coefficient of friction on the floor, and μ' that on the wall.

1315. (Editor.)—Divide unity into four parts such that, if the square of one of them be diminished by four times the product of the other three, the remainder may be a rational square.

1334. (Dr. Rutherford, F.R.A.S.)—A heavy uniform beam (AB) moves freely about a hinge at A, and an elastic string is attached to the extremity B, and fixed at a point C in the same horizontal line as A, at a distance (AC) equal to the length of the beam. The natural length of the elastic string is equal to half that of the beam, and its elasticity is such that the weight of the beam would stretch it to twice its natural length. Find the angle which the string makes with the horizon when the system is in equilibrium.

1340. (W. S. B. Woolhouse, F.R.A.S., F.S.S.)—Determine the value of the expression

$$(-1)^{\frac{1}{2}} \sin\left[(-1)^{\frac{1}{2}} \log_e \left\{x+(x^2+1)^{\frac{1}{2}}\right\}\right] \cos\left[(-1)^{\frac{1}{2}} \log_e \left\{x+(x^2-1)^{\frac{1}{2}}\right\}\right].$$

1369. (* * .)—A telegraph wire capable of sustaining a tension of a pounds, and weighing b pounds to the yard, is stretched, as tightly as is consistent with its not breaking, over a series of posts. The posts are generally equally distant from one another, c yards of wire hanging between every successive two; but, the telegraph having to cross a river which is more than c yards wide, a length of d yards of wire hangs between the posts on each bank. Show that, in order that these two posts may have no tendency to break, each must be inclined to the vertical at an angle equal to one-half of $\sin^{-1}(bd/2a) - \sin^{-1}(bc/2a)$, the wire being supposed to be perfectly flexible, and the tops of all the posts to be in the same horizontal straight line.

1451. (Dr. Hirst, F.R.S.)—Prove that if

$$F(x,\ y,\ a,\ b,\ c,\ f,\ g,\ h) = 0$$

be the equation of the n^{th} Pedal of the conic

$$(a,\ b,\ c,\ f,\ g,\ h)\,(x,\ y,\ 1)^2 = 0,$$

the equation of the $(-n+1)^{\text{th}}$ Pedal will be

$$F\left(-\frac{x}{r^2},\ -\frac{y}{r^2},\ \frac{d\Delta}{da},\ \frac{d\Delta}{db},\ \frac{d\Delta}{dc},\ \tfrac{1}{2}\frac{d\Delta}{df},\ \tfrac{1}{2}\frac{d\Delta}{dg},\ \tfrac{1}{2}\frac{d\Delta}{dh}\right) = 0,$$

where $r^2 = x^2 + y^2$, and $\Delta = \begin{vmatrix} a\,h\,g \\ h\,b\,f \\ g\,f\ \ \end{vmatrix}$.

1471. (Dr. Hirst, F.R.S.)—Prove the following theorems:—

1. The primitive being a curve of the n^{th} order, passing α times through the origin, and circular in the order β; its inverse is of the order $2n-\alpha-2\beta$, passes $n-2\beta$ times through the origin, and is circular in the order $n-\alpha-\beta$.

2. The first negative pedal of the primitive will, consequently, be of the $(2n-\alpha-2\beta)^{\text{th}}$ class, will touch the line at infinity $n-2\beta$ times, and have a focus at the origin of the order $n-\alpha-\beta$. Its order will, in general, be $n(n+1)-\alpha(\alpha+1)-2\beta(\beta+1)$; subject, however, to reduction whenever the primitive has other multiple points, itself touches the line at infinity, or has a focus at the origin.

3. The primitive being a curve of the n^{th} class, touching the line at infinity α times, and having a focus at the origin of the order β, its first positive pedal is a curve of the order $2n-\alpha-2\beta$, passing $n-2\beta$ times through the origin, and circular in the order $n-\alpha-\beta$. Its class will consequently be $n(n+1)-\alpha(\alpha+1)-2\beta(\beta+1)$; subject, however, to reduction whenever the primitive is itself circular, passes through the origin, or has other multiple tangents.

[A curve is said to be circular in the order β when it passes β times through each of the two imaginary circular points at infinity; and when it is touched β times by each of the lines joining these circular points to the origin, it is said to have, at the latter point, a focus of the β^{th} order.]

1487. (T. T. Wilkinson, F.R.A.S.)—The distance of two horizontal points is 20 inches; to these points the ends of two strings, each 12 inches long, are attached; their other ends are fastened to two uniform heavy rods, which revolve freely round a hinge at the other extremity. Required the angle which the rods make with each other when at rest.

1610. (Dr. Booth, F.R.S.)—The polar equations of the circle and of a curve, which I have named the Logocyclic Curve, are

$$r^2 - 2ar\cos\theta + a^2 = 0, \quad r^2 - 2ar\sec\theta + a^2 = 0.$$

From these equations establish a few of the leading analogies which exist between *circular* and *parabolic* trigonometry.

2572. (Professor Sylvester.)—(1) If $h,\ k,\ l,\ \ldots$, are the distinct prime factors of n, prove that the number of points (which I term pluperfect points of the n^{th} order) at which a cubic curve can have the highest degree of contact with a curve of the degree n (not composed of repetitions of a curve of lower degree) is $9n^2(1-h^{-2})(1-k^{-2})(1-l^{-2})$.

(2) Show that if the first tree in the solution to Quest. 2473 (see Vol. VIII., p. 106) be planted at a pluperfect point of the nth order in a cubic, the sequence of tree-marks 1, 2, 4, 5, 7, ... may be replaced by recurring periods of $2n$ numerals, and that the two halves of each period will consist of the same $2n$ numerals arranged in reverse order, that in fact only the first n of the numerals 1, 2, 4, ... need appear in the result.

(3) Hence prove that n trees may be so arranged as to contain between them $E\left\{\frac{1}{6}n(n-1)\right\} - E\left\{\frac{1}{3}n\right\}$ rows of three in a row, where E (the symbol of entirety) denotes that the integer part only is to be taken of the function which it governs.

[Thus, for 81 trees the number will be 1053 instead of 800, the number obtained when the first tree is at a non-pluperfect point; and for 15 trees the number is 30 instead of 26, the number stated in the Editorial reference at the end of the solution above cited as applicable to 15 points, but which is in fact, the number for 14, according to the formula stated above. It must, however, be observed that this formula does not in general give the absolute maximum. Thus, for $n = 9$ the formula gives only 9 rows, whereas the true maximum (dealing with *real* trees) is 10. In fact, $E\left\{\frac{1}{6}n(n-1)\right\}$ is the arithmetical limit obtained by dividing the number of duads of n by 3, the number of them in each row. Thus, $E\left\{\frac{1}{3}n\right\}$ is the difference between this limit and the actual number of the formula, which may also be expressed as equal to $\frac{1}{8}(n^2-3n)$ when this is an integer, and the integer immediately superior to it when it is fractional. Thus we see that with 1000 trees, the number of rows containing three in a row may be made equal to 166167. In a word, when the first tree is planted at an ordinary point in the cubic, the number of rows given by the formulæ in Quest. 2473 is $E\left\{\frac{1}{8}(u-1)^2\right\}$, when at a pluperfect point the number is

$$E\left\{\tfrac{1}{6}(n-1)(n-2)\right\}.$$

Another method of placing points on a cubic will give 10 rows for 9 points, and possibly in all cases the true maximum.]

2581. (M. Collins, B.A.)—In an ellipse, find (1) the locus of the middle points of all the normals, (2) the locus of the poles of the normals, (3) the minimum normal chord, and (4) the normal chord that cuts off the least elliptic segment.

2588. (W. B. Davis, B.A.)—Divide 126 consecutive numbers into groups of 7, so that there may be six pairs of consecutive numbers and six pairs only, distributed amongst the groups. [Each group of 7 when added together will make a multiple of 127, which will serve for verification.]

2589. (Professor Sylvester.)—Let a tree springing from the ground at A rise at B, thence bifurcate into BC, BD, again bifurcate at C and at D, and so on for any number n of times, thus giving rise to $1+1+2+4+\ldots+2^{n-1}$, that is 2^n points of salience (A included); prove that the ramification may be constructed subjected to the condition that the right line connecting any two points of salience not springing immediately one from the other shall pass through a third point of salience.

[By the points of salience are meant the joints where bifurcation takes place and the point where the tree rises from the ground.]

2599. (W. S. B. Woolhouse, F.R.A.S.)—Each set of simultaneous roots of a given linear equation $A_1z_1 + A_2z_2 + \dots A_nz_n = B$ is arranged in the order of magnitude; find the average value of the root which stands the mth in order.

2610. (Professor Sylvester.)—The point of intersection of two right lines, and also two other points on each of them being given as five of the flexures of a cubic curve, required to determine the locus of the remaining four flexures.

2613. (S. Watson.)—Three points are taken at random, one on each side of a given triangle; find the average area of the circle drawn through them.

2625. (T. Cotterill, M.A.)—Conicoids circumscribing a quartic curve in space have a common self-conjugate tetrahedron. Show that a plane cuts the quartic in four points, and the tetrahedron in four lines, such that triangles can be found circumscribing any conic inscribed in the four lines conjugate to any conic through the four points.

2633. (Professor Tait.)—Show that the greatest amount of mechanical effect which can be obtained from a set of unequally heated equal and similar masses)whose specific heat does not vary with their temperature) is proportional to the excess of the arithmetic over the geometric mean of their absolute temperatures.

2634. (A. Crum Brown, D.Sc.) — $4m + 2n$ separate strings, having each a black and a white end, are taken. $4m$ of these are united in groups of 4 by their white ends, and the white ends of all the others are to be left free. In how many ways may the black ends of the system be united two and two so as to form a continuous aggregate? It is, of course, contemplated that two black ends belonging to the same group of four may be united. This gives rise to the further question:—Divide the above arrangements into classes according to the number of groups of four, each of which has two of its black ends united.

2635. (Professor Cayley.)—Find the equation of the surface which is the envelope of the quadric surface $ax^2 + by^2 + cz^2 + dw^2 = 0$, where a, b, c, d are variable parameters connected by the equation

$$Abc + Bca + Cab + Fad + Gbd + Hcd = 0,$$

and consider in particular the case in which the constants A, B, C, F, G, H satisfy the condition $(AF)^{\frac{1}{2}} + (BG)^{\frac{1}{2}} + (CH)^{\frac{1}{2}} = 0$.

2637. (Professor Whitworth.) — If ϵ_r^x denote the sum of the series obtained by expanding ϵ^x in ascending powers of x as far as the term involving x^r inclusive, then

$$1 = \epsilon_n^{-1} + \frac{\epsilon_{n-1}^{-1}}{1} + \frac{\epsilon_{n-2}^{-1}}{1.2} + \frac{\epsilon_{n-3}^{-1}}{1.2.3} + \dots + \frac{\epsilon_0^{-1}}{n!}.$$

2739. (M. W. Crofton, F.R.S.)—(1) Two points are taken at random within a circle, and a random straight line crosses the circle; find the probability that it shall pass between the points. (2) A straight line taken at random crosses any convex area; let p_1 be the probability that it passes between two points taken at random in the area; again, let p_2 be the probability that it meets the triangle formed by three points taken at random in the area; then $p_2 = \frac{3}{2}p_1$.

2750. (Artemas Martin.)—Three equal circles, each 4 inches in diameter, are drawn at random on a circular slate whose diameter is 12 inches; find the probability that each circle intersects the other two.

2751. (J. Wilson.)—Draw a tangent to a given circle so that its intercepts on two lines given in position may (1) have a given ratio; or so that (2) their sum, or (3) their difference, or (4) their rectangle, or (5) the sum of their squares, or (6) the difference of their squares, may be given, a maximum or a minimum.

2758. (Professor Sylvester.)—(1) Prove that the curve designated by the polar equation $\theta = 2\left\{(a \pm r)/a\right\}^{\frac{1}{2}} - \tan^{-1}\left[\frac{1}{4} - \left\{a/(a \pm r)\right\}^{\frac{1}{2}}\right]$ has for its *fourth* evolute a circle with centre at the origin and radius $= \frac{3}{2}a$. (2) Find also the polar equation to its second evolute.

2759. (Professor Hirst.)—On two given lines AB, CD construct (1) two similar circle-segments whose arcs shall touch one another; and find (2) how many solutions there are.

2775. (J. Wilson.)—Prove that the locus of a point, the square of the tangent from which to a fixed circle varies as its distance from a fixed line, is a circle which cuts, does not cut, or touches, the fixed line, according as it cuts the circle, passes without it, or touches it.

2777. (Professor Crofton, F.R.S.)—If a bicircular quartic pass through four fixed points on a circle, and have one focus at the centre of the circle, the locus of its three remaining foci consists of the two circular cubics whose foci are the four given points [the same locus as in Questions 1990 and 2332]. If the given focus be not at the centre of the circle, the locus of the three others will generally be the two bicircular quartics passing through that focus, and having the four given points as foci.

2785. (W. S. Burnside, M.A.)—Assuming

$$(a, \beta, \gamma)(x, y)^2 \equiv X, \quad (a', \beta', \gamma')(x, y)^2 \equiv Y, \quad (a'', \beta'', \gamma'')(x, y)^2 \equiv Z,$$

the binary quartic $(a, b, c, d, e)(x, y)^4 \equiv U$ may be replaced by a ternary quadric V with an auxiliary quadric $W = 0$.

(1) Show that the ordinary methods of solving the quartic equation $U = 0$ may be made to depend on the reduction of U and V or $U + \lambda V$ to special forms, and actually form in this manner the reducing cubic of the quartic, namely, $$\mu^3 - S\mu + 2T = 0,$$
where $$S \equiv ac - 4bd + 3c^2, \quad T \equiv ace + 2bcd - ad^2 - eb^2 - c^3.$$

(2) We may derive a geometrical construction for the roots of the Hessian of U by means of the harmonic conic of V and W.

(3) Find the condition that the conics $V = 0$ and $W = 0$ should touch, and thus derive the discriminant of the quartic U.

2788. (N'Importe.)—The vertex of an isosceles triangle moves along a fixed diameter of a parabola, *one* of the two equal sides passes through the vertex of the parabola, and *the other* is an ordinate to the fixed diameter; find the locus of the foot of the perpendicular drawn from the vertex of the parabola on the base of the triangle. Trace the curve for every possible variety of position of the moving triangle.

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